Proofs of Conjectures in Aggregating Inconsistent Information: Ranking and Clustering

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Abstract

In [ACN], Ailon, Charikar and Newman address the problems of rank aggregation, minimum feedback arc set in tournaments, correlation clustering and consensus clustering. They present new and improved combinatorial algorithms for approximating these problems. They also present variants of these algorithms based on linear programming rounding techniques, further improving the approximation factors. The LP based results in [ACN] are, however, left as conjectures based on numerical evidence. In this work, which should be read as an annex to [ACN], we prove these conjectures and henceforth establish theorems.

0 Introduction

We refer the reader to [ACN] for the definitions of the problems (weighted) FAS-TOURNAMENT, RANK-AGGREGATION, (weighted) CORRELATION-CLUSTERING and CONSENSUS-CLUSTERING, as well as the definitions of the algorithms FASLP-PIVOT, PICK-A-PERM, CCLP-PIVOT and PICK-A-CLUSTER. We prove the following theorems.

Theorem 1 *The best of* FASLP-PIVOT *and* PICK-A-PERM *on* RANK-AGGREGATION *is a* $4/3$ *approximation.*

Theorem 2 FASLP-PIVOT *on weighted* FAS-TOURNAMENT *with probability constraints is a* $5/2$ *approximation.*

Theorem 3 FASLP-PIVOT *on weighted* FAS-TOURNAMENT *with triangle inequality and probability constraints is a* 2 *approximation.*

Theorem 4 *The best of* CCLP-PIVOT *and* PICK-A-CLUSTER *on* CONSENSUS-CLUSTERING *is a* $4/3$ *approximation.*

Theorem 5 CCLP-PIVOT *on weighted* CORRELATION-CLUSTERING *with probability constraints is a* $5/2$ *approximation.*

Theorem 6 CCLP-PIVOT *on weighted* CORRELATION-CLUSTERING *with triangle inequality and probability constraints is a* 2 *approximation.*

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The proofs of Theorems 1-6 in Sections 1-6 are based on showing that certain multinomials are nonpositive in certain polytopes. The reader is again referred to [ACN] for the reduction of the theorems to the multinomial inequalities.

1 Proof of Theorem 1

Define

$$\begin{aligned} piv(x_1,x_2,x_3,w_1,w_2,w_3) &= x_1x_2w_3 + (1-x_1)(1-x_2)(1-w_3) \\ &+ x_2x_3w_1 + (1-x_2)(1-x_3)(1-w_1) \\ &+ x_3x_1w_2 + (1-x_3)(1-x_1)(1-w_2) \end{aligned}$$

$$\begin{aligned} pap(x_1,x_2,x_3,w_1,w_2,w_3) &= (x_1x_2 + (1-x_1)(1-x_2))2w_3(1-w_3) \\ &+ (x_2x_3 + (1-x_2)(1-x_3))2w_1(1-w_1) \\ &+ (x_3x_1 + (1-x_3)(1-x_1))2w_2(1-w_2) \end{aligned} \tag{1}$$

$$\begin{aligned} opt(x_1,x_2,x_3,w_1,w_2,w_3) &= (x_1x_2 + (1-x_1)(1-x_2))(x_3(1-w_3) + (1-x_3)w_3) \\ &+ (x_2x_3 + (1-x_2)(1-x_3))(x_1(1-w_1) + (1-x_1)w_1) \\ &+ (x_3x_1 + (1-x_3)(1-x_1))(x_2(1-w_2) + (1-x_2)w_2) \end{aligned}$$

$$f(x_1,x_2,x_3,w_1,w_2,w_3) = \frac{2}{3}piv + \frac{1}{3}pap - \frac{4}{3}opt$$

We want to prove that $f \le 0$ in the following polytope:

$$\begin{aligned} 0 \le w_i \le 1 \quad i = 1,2,3 \\ 0 \le x_i \le 1 \quad i = 1,2,3 \\ 1 \le w_1 + w_2 + w_3 \le 2 \\ 1 \le x_1 + x_2 + x_3 \le 2 \end{aligned} \tag{2}$$

To do this, we first make the following assumption, without loss of generality

$$w_1 + w_2 + w_3 \ge 3/2.$$

(Otherwise, we can replace all w_i with $1 - w_i$ and all x_i with $1 - x_i$). Next, we assume a slightly larger polytope for the x_i variables, therefore proving something slightly stronger. More precisely, we prove that $f \le 0$ in the following polytope

$$\begin{aligned} 0 \le w_i \le 1 \quad i = 1,2,3 \\ 0 \le x_i \le 1 \quad i = 1,2,3 \\ 3/2 \le w_1 + w_2 + w_3 \le 2 \\ x_1 + x_2 + x_3 \le 2 \end{aligned} \tag{3}$$

(we removed the constraint $1 \le x_1 + x_2 + x_3$). Let P_x denote the x polytope, and P_w denote the w polytope, so the region of feasibility is $P_x \times P_w \subseteq \mathbf{R}^6$.

We first observe that for any $i = 1, 2, 3$, f if linear in x_i when all other variables x_j, w_k are fixed, for $j \in \{1, 2, 3\} - \{i\}$, $k \in \{1, 2, 3\}$. Therefore, if at a certain point $(\mathbf{x}, \mathbf{w}) = (x_1, x_2, x_3, w_1, w_2, w_3)$ it is possible to both increase and decrease x_i without changing the other variables and without leaving the polytope, then $(\mathbf{x}, \mathbf{w})$ is not a local maximum of f. Therefore, it suffices to analyze f in the following regions:

$$\begin{aligned} P_1 &= \{(0, 0, 0)\} \times P_w \\ P_2 &= \{(0, 0, 1)\} \times P_w \\ P_3 &= \{(0, 1, 0)\} \times P_w \\ P_4 &= \{(1, 0, 0)\} \times P_w \\ P_5 &= (P_x \cap \{x_1 + x_2 + x_3 = 2\}) \times P_w \end{aligned} \tag{4}$$

We start with P_1. Simple calculations show that

$$f(0, 0, 0, w_1, w_2, w_3) = 2 - \frac{4}{3}(w_1 + w_2 + w_3) - \frac{2}{3}(w_1^2 + w_2^2 + w_3^2) \ .$$

Clearly, this cannot be more than 0 under our assumption that $w_1 + w_2 + w_3 \geq 3/2$.

Next, we analyze f on P_2. Simple calculations show that

$$f(1, 0, 0, w_1, w_2, w_3) = -\frac{2}{3}(w_1 - 1)^2,$$

which is, again, clearly at most 0.

Regions P_3 and P_4 are symmetric to P_2 under variable renaming.

It remains to analyze f on P_5. To do this we will slightly enlarge P_5 and replace it with P_5', defined as

$$P_5' = (P_x \cap \{x_1 + x_2 + x_3 = 2\}) \times P_w',$$

where P_w' is the region obtained when removing the constraint $3/2 \leq w_1 + w_2 + w_3$ from P_w. We now write the partial derivatives of f w.r.t. w_3 (similarly for w_1, w_2):

$$\frac{\partial f}{\partial w_3} = \frac{1}{3}((8x_3 - 4w_3)(x_1x_2 + (1 - x_1)(1 - x_2)) - 4(1 - x_1)(1 - x_2)) \ . \tag{5}$$

Elementary computation shows that $\bigtriangledown_w f = 0$ exactly when

$$\begin{aligned} w_1 &= \frac{x_2x_3}{x_2x_3 + (1 - x_2)(1 - x_3)} + 2x_1 - 1 \\ w_2 &= \frac{x_3x_1}{x_3x_1 + (1 - x_3)(1 - x_1)} + 2x_2 - 1 \\ w_3 &= \frac{x_1x_2}{x_1x_2 + (1 - x_1)(1 - x_2)} + 2x_3 - 1 \end{aligned} \tag{6}$$

Because we assume $x_1 + x_2 + x_3 = 2$, we have that for any $i \neq j$, $x_i + x_j \geq 1$, equivalently, $x_ix_j \geq (1 - x_i)(1 - x_j)$. Therefore (6) implies the following for $i = 1, 2, 3$:

$$w_i \geq \frac{1}{2} + 2x_i - 1 \ . \tag{7}$$

Consequently, $w_1 + w_2 + w_3 \geq \frac{3}{2} + 2(x_1 + x_2 + x_3) - 3 \geq \frac{5}{2} > 2$, which is outside P_w.

Similarly, setting $w_1 = 1$ and requiring that $\partial f/\partial w_2 = \partial f/\partial w_3 = 0$ results in a point which is outside P_w, unless $w_2 + w_3 = 1$ (we take care of this case $w_1 + w_2 + w_3 = 2$ shortly). . The same for $w_2 = 1$ or $w_3 = 1$. If two of w_1, w_2, w_3 are 1, then the third is 0, which is again the case $w_1 + w_2 + w_3 = 2$.

We now show that no local maximum of f can be obtained in P'_5 when $w_i = 0$, for any i. Assume $i = 1$ (similarly for $i = 2, 3$). Using elementary techniques (details omitted), we can show that $\partial f/\partial w_1 \geq 0$ when $w_1 = 0$. Therefore, if $(w_2, w_3) \neq (1, 1)$, then slightly increasing w_1 would increase f. If, however, $(w_1, w_2, w_3) = (0, 1, 1)$, then again, using elementary techniques, we show that $f \leq 0$.

It therefore remains to investigate f on the region

$$(P_x \cap \{x_1 + x_2 + x_3 = 2\}) \times (P_w \cap \{w_1 + w_2 + w_3 = 2\}).$$

Let $Q_x = P_x \cap \{x_1 + x_2 + x_3 = 2\}$ and $Q_w = P_w \cap \{w_1 + w_2 + w_3 = 2\}$. Instead of investigating f on $Q_x \times Q_w$, we will investigate it on $Q_x \times H$, where H is the affince closure of Q_w in $\mathbf{R}^3$ (the unique $2-\textit{dimensional}$ hyperplane containing Q_w). We fix $\mathbf{x} \in Q_x$ and investigate f as a function of $\mathbf{w}$. To do this, we find the global maximum of f in H. It is easy to verify that unless $\mathbf{x}$ is a vertex of Q_x (a case which we already took care of), $\lim_{\mathbf{w}\to\infty} f = -\infty$, therefore f has a global maximum on $Q_x \times H$. This maximum is obtained when

$$\bigtriangledown_w f = \lambda(1, 1, 1)$$

for some $\lambda \neq 0$ (we already took care of the case $\lambda = 0$). That is, $\mathbf{w}$ has to satisfy:

$$\frac{\partial f}{\partial w_1} = \frac{\partial f}{\partial w_2} = \frac{\partial f}{\partial w_3} \tag{8}$$

on H. It is immediate to verify that the unique solution to (8) is $\mathbf{w} = \mathbf{x}$. It therefore remains to investigate $g(\mathbf{x}) = f(\mathbf{x}, \mathbf{x})$ on Q_x. We start by investigating it on $intQ_x$. A necessary condition for a local maximum of g on $intQ_x$ is

$$\bigtriangledown g = \lambda'(1, 1, 1),$$

for some λ'. If $\lambda' = 0$, then $\bigtriangledown g = 0$, which solves to the following possibilities: $(x_1, x_2, x_3) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. On all of these possibilities it is elementary to verify that $g \leq 0$. If $\lambda' \neq 0$, it means that

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial x_3} .$$

Elementary calculations shows that this happens when

$$(x_1, x_2, x_3) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (2/3, 2/3, 2/3)\} .$$

It is immediate to verify that $g \leq 0$ on all of these possibilities. We now investigate g on the edges of Q_x. We pick the edge $int(\{(1, x_2, 1 - x_2\} \cap Q_x)$. (other 2 edges argued similarly). Letting $h(x_2) = g(1, x_2, 1 - x_2)$, we immediately verify that $h \equiv 0$. The vertices of Q_x can be argued for by trying all possibilities and verifying that $g \leq 0$. □

2 Proof of Theorem 2

Let piv, opt be defined as in Section 1. Let

$$f(\mathbf{x}, \mathbf{w}) = piv - \frac{5}{2} opt ,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. We prove that $f \le 0$ on

$$\begin{aligned} &0 \le w_i \le 1 \quad i = 1,2,3 \\ &0 \le x_i \le 1 \quad i = 1,2,3 \\ &1 \le x_1 + x_2 + x_3 \le 2 \ . \end{aligned} \tag{9}$$

Let $P_{\mathbf{x}}$ denote the polytope corresponding to $\mathbf{x}$ and P_w denote the polytope (cube) corresponding to $\mathbf{w}$.

Clearly f is linear in $\mathbf{w}$ when $\mathbf{x}$ is fixed. So it suffices to investigate f on the vertices of $P_{\mathbf{w}}$. Due to the symmetric structure of f, it will suffice to investigate f on two points $\mathbf{w} = (0,0,0)$ and $\mathbf{w} = (0,0,1)$.

1. Case $\mathbf{w} = (0,0,0)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 0, 0, 0)$. So

   $$g = 3 - \frac{9}{2}(x_1 + x_2 + x_3) + 6(x_1x_2 + x_2x_3 + x_3x_1) - 15x_1x_2x_3 \ .$$

   Since g is linear in each one of x_i when the others are fixed, we can assume that either $x_1 + x_2 + x_3 = 1$ or $x_1 + x_2 + x_3 = 2$ (otherwise, we can slightly both increase and decrease one of the x_i's without leaving the polytope, therefore we are not at a local maximum).

   (a) Subcase $x_1 + x_2 + x_3 = 1$. Let $h(x_1, x_2) = g(x_1, x_2, 1 - x_1 - x_2)$. We want to show that $h \le 0$ on $0 \le x_1, 0 \le x_2, x_1 + x_2 \le 1$.

   $$h(x_1, x_2) = -\frac{3}{2} + 6(x_1 + x_2) - 6(x_1^2 + x_2^2) + 15(x_1x_2^2 + x_1^2x_2) - 21x_1x_2 \ .$$

   If $x_1 = 0$ or $x_1 = 1 - x_2$ then $h = -3(1 - 2x_2)^2/2 \le 0$. If $\partial h/\partial x_1 = 0$ then $x_1 = (1 - x_2)/2$ and $h = -3x_2(1 - 4x_2 + 5x_2^2)/4$ which is ≤ 0 on $x_2 \in [0, 1]$.

   (b) Subcase $x_1 + x_2 + x_3 = 2$. Let $h(x_1, x_2) = g(x_1, x_2, 2 - x_1 - x_2)$. We want to show that $h \le 0$ on $x_1 \le 1, x_2 \le 1, x_1 + x_2 \ge 1$.

   $$h(x_1, x_2) = -6 + 12(x_1 + x_2) - 6(x_1^2 + x_2^2) + 15(x_1^2x_2 + x_1x_2^2) - 36x_1x_2 \ .$$

   If $x_1 = 1$ or $x_1 = 1 - x_2$ then $h = -9(1 - x_2)x_2 \le 0$ on $x_2 \in [0, 1]$. If $\partial h/\partial x_1 = 0$ then $x_1 = (2 - x_2)/2$ and $h = -3x_2(12 - 14x_2 + 5x_2^2)/4$, which is ≤ 0 when $x_2 \in [0, 1]$.

2. Case $\mathbf{w} = (0,0,1)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 0, 0, 1)$. So

   $$g = -\frac{1}{2} - (x_1 + x_2) + \frac{1}{2}x_3 + (x_1x_2 + x_2x_3 + x_3x_1) - 5x_1x_2x_3 \ .$$

   As before, it suffices to investigate g when either $x_1 + x_2 + x_3 = 1$ or $x_1 + x_2 + x_3 = 2$.

   (a) Subcase $x_1 + x_2 + x_3 = 1$. Let $h(x_1, x_2) = g(x_1, x_2, 1 - x_1 - x_2)$. We want to show that $h \le 0$ on $0 \le x_1, 0 \le x_2, x_1 + x_2 \le 1$.

   $$h(x_1, x_2) = -\frac{1}{2}(x_1 + x_2) - (x_1^2 + x_2^2) + 5(x_1x_2^2 + x_1^2x_2) - 6x_1x_2 \ .$$

   If $x_1 = 0$ then $h = -\frac{1}{2}x_2(1 + 2x_2) \le 0$ on $x_2 \in [0, 1]$. Similarly holds when $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -\frac{3}{2} + x_2 - x_2^2 \le 0$. We completed checking the boundary. There is no solution to $\nabla_{x_1, x_2} h = 0$ in the interesting region.

(b) Subcase $x_1 + x_2 + x_3 = 2$. Let $h(x_1, x_2) = g(x_1, x_2, 2 - x_1 - x_2)$. We want to show that $h \leq 0$ on $x_1 \leq 1, x_2 \leq 1, x_1 + x_2 \geq 1$.

$$h(x_1, x_2) = \frac{1}{2} + \frac{1}{2}(x_1 + x_2) - (x_1^2 + x_2^2) + 5(x_1^2 x_2 + x_1 x_2^2) - 11 x_1 x_2 \ .$$

If $x_1 = 1$ then $h = x_2(-11 + 8x_2)/2 \leq 0$ on $x_2 \in [0, 1]$. Similarly if $x_2 = 1$. If $x_1 = 1 - x_2$ then $h = -4(1-x_2)x_2 \leq 0$ on $x_2 \in [0, 1]$. The only solution to $\bigtriangledown_{x_1,x_2} h = 0$ in the interesting region is $x_1 = x_2 = (13 + \sqrt{139})/30$. At that point $h = (-937 - 139\sqrt{139})/1350 \leq 0$.

$\square$

3 Proof of Theorem 3

Let piv, opt be defined as in Section 1. Let

$$f(\mathbf{x}, \mathbf{w}) = piv - 2opt \ ,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. We prove that $f \leq 0$ on

$$\begin{aligned} &0 \leq w_i \leq 1 \quad i = 1, 2, 3 \\ &0 \leq x_i \leq 1 \quad i = 1, 2, 3 \\ &1 \leq w_1 + w_2 + w_3 \leq 2 \\ &1 \leq x_1 + x_2 + x_3 \leq 2 \ . \end{aligned} \tag{10}$$

Let $P_\mathbf{x}$ denote the polytope corresponding to $\mathbf{x}$ and P_w denote the polytope corresponding to $\mathbf{w}$.

Clearly f is linear in $\mathbf{w}$ when $\mathbf{x}$ is fixed. So it suffices to investigate f on the vertices of $P_\mathbf{w}$. Due to the symmetric structure of f, it will suffice to investigate f on the point $\mathbf{w} = (0, 0, 1)$.

Let $g(\mathbf{x}) = f(\mathbf{x}, 0, 0, 1)$. So

$$g = -(x_1 + x_2) + (x_1 x_2 + x_2 x_3 + x_3 x_1) - 4 x_1 x_2 x_3 \ .$$

As in the previous sections, it suffices to investigate g when either $x_1 + x_2 + x_3 = 1$ or $x_1 + x_2 + x_3 = 2$.

1. Case $x_1 + x_2 + x_3 = 1$. Let $h(x_1, x_2) = g(x_1, x_2, 1 - x_1 - x_2)$. We want to show that $h \leq 0$ on $0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1$.

$$h(x_1, x_2) = -(x_1^2 + x_2^2) + 4(x_1 x_2^2 + x_1^2 x_2) - 5 x_1 x_2 \ .$$

If $x_1 = 0$ then $h = -x_2^2 \leq 0$. Similarly holds when $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -1 + x_2 - x_2^2$, which is ≤ 0 when $x2 \in [0, 1]$. We completed checking the boundary. The only solutions to $\bigtriangledown_{x_1,x_2} h = 0$ in the interior of the interesting region are $x_1 = x_2 = 7/12$ and $x_1 = x_2 = 0$. In the former case we get $h = -343/432 \leq 0$, and in the latter we get $h = 0$.

2. Case $x_1 + x_2 + x_3 = 2$. Let $h(x_1, x_2) = g(x_1, x_2, 2 - x_1 - x_2)$. We want to show that $h \leq 0$ on $x_1 \leq 1, x_2 \leq 1, x_1 + x_2 \geq 1$.

$$h(x_1, x_2) = (x_1 + x_2) - (x_1^2 + x_2^2) + 4(x_1^2 x_2 + x_1 x_2^2) - 9 x_1 x_2 \ .$$

If $x_1 = 1$ then $h = x_2(-4 + 3x_2) \leq 0$ on $x_2 \in [0, 1]$. Similarly if $x_2 = 1$. If $x_1 = 1 - x_2$ then $h = -3(1 - x_2)x_2 \leq 0$ on $x_2 \in [0, 1]$. The only solution to $\bigtriangledown_{x_1,x_2} h = 0$ in the interesting region is $x_1 = x_2 = (11 + \sqrt{73})/24$. At that point $h = (-539 - 73\sqrt{73})/864 \leq 0$.

$\square$

4 Proof of Theorem 4

Define

$$\begin{aligned}
piv(x_1,x_2,x_3,w_1,w_2,w_3) &= (1-x_1)(1-x_2)w_3 + (x_1(1-x_2)+(1-x_1)x_2)(1-w_3) \\
&+ (1-x_2)(1-x_3)w_1 + (x_2(1-x_3)+(1-x_2)x_3)(1-w_1) \\
&+ (1-x_3)(1-x_1)w_2 + (x_3(1-x_1)+(1-x_3)x_1)(1-w_2)
\end{aligned}$$

$$\begin{aligned}
pap(x_1,x_2,x_3,w_1,w_2,w_3) &= ((1-x_1)(1-x_2) + x_1(1-x_2)+(1-x_1)x_2)2w_3(1-w_3) \\
&+ ((1-x_2)(1-x_3) + x_2(1-x_3)+(1-x_2)x_3)2w_1(1-w_1) \\
&+ ((1-x_3)(1-x_1) + x_3(1-x_1)+(1-x_3)x_1)2w_2(1-w_2)
\end{aligned}$$

$$\begin{aligned}
opt(x_1,x_2,x_3,w_1,w_2,w_3) &= ((1-x_1)(1-x_2) + x_1(1-x_2)+(1-x_1)x_2)(x_3(1-w_3)+(1-x_3)w_3) \\
&+ ((1-x_2)(1-x_3) + x_2(1-x_3)+(1-x_2)x_3)(x_1(1-w_1)+(1-x_1)w_1) \\
&+ ((1-x_3)(1-x_1) + x_3(1-x_1)+(1-x_3)x_1)(x_2(1-w_2)+(1-x_2)w_2)
\end{aligned}$$

$$f(x_1,x_2,x_3,w_1,w_2,w_3) = \frac{2}{3}piv + \frac{1}{3}pap - \frac{4}{3}opt \tag{11}$$

We want to prove that $f \le 0$ in the following polytope:

$$\begin{aligned}
&0 \le w_i \le 1 \quad i = 1,2,3 \\
&0 \le x_i \le 1 \quad i = 1,2,3 \\
&w_1 \le w_2 + w_3,\ w_2 \le w_3 + w_1,\ w_3 \le w_1 + w_2 \\
&x_1 \le x_2 + x_3,\ x_2 \le x_3 + x_1,\ x_3 \le x_1 + x_2
\end{aligned} \tag{12}$$

Let P_x denote the x polytope, and P_w denote the w polytope, so the region of feasibility is $P_x \times P_w \subseteq \mathbf{R}^6$. It is not hard to see that

$$P_x = P_w = \text{conv}\{(0,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}\ .$$

We first observe that for any $i = 1,2,3$, f if linear in x_i when all other variables x_j, w_k are fixed, for $j \in \{1,2,3\} - \{i\}$, $k \in \{1,2,3\}$. Therefore, if at a certain point $(\mathbf{x},\mathbf{w}) = (x_1,x_2,x_3,w_1,w_2,w_3)$ it is possible to both increase and decrease x_i without changing the other variables and without leaving the polytope, then $(\mathbf{x},\mathbf{w})$ is not a local maximum of f. Therefore, it suffices to analyze f in the following regions:

$$\begin{aligned}
P_1 &= \{(1,1,1)\} \times P_w \\
P_2 &= \text{conv}\{(0,0,0),(0,1,1),(1,0,1)\} \times P_w \\
P_3 &= \text{conv}\{(0,0,0),(1,0,1),(1,1,0)\} \times P_w \\
P_4 &= \text{conv}\{(0,0,0),(0,1,1),(1,1,0)\} \times P_w
\end{aligned} \tag{13}$$

It is easy to show that $f(1,1,1,w_1,w_2,w_3) = 0$. It remains to analyze f on P_2,P_3 and P_4. Due to symmetery, we can consider only P_2, which we rewrite as:

$$P_2 = (P_x \cap \{x_3 = x_1 + x_2\}) \times P_w\ .$$

From now on we therefore assume $x_3 = x_1 + x_2$.

$$\frac{\partial f}{\partial w_3} = \frac{1}{3}\left[(8x_3 - 4w_3)((1-x_1)(1-x_2) + x1(1-x_2) + x2(1-x_1)) - 4(x1(1-x_2) + (1-x_1)x2)\right] . \tag{14}$$

(Similarly for $\partial f/\partial w_1, \partial f/\partial w_2$). Elementary computation shows that $\bigtriangledown_{\mathbf{w}} f = 0$ exactly when

$$\begin{aligned} w_1 = w_1^* &:= \frac{(1-x_2)(1-x_3)}{(1-x_2)(1-x_3) + x2(1-x_3) + (1-x_2)x_3} + 2x_1 - 1 \\ w_2 = w_2^* &:= \frac{(1-x_3)(1-x_1)}{(1-x_3)(1-x_1) + x3(1-x_1) + (1-x_3)x_1} + 2x_2 - 1 \\ w_3 = w_3^* &:= \frac{(1-x_1)(1-x_2)}{(1-x_1)(1-x_2) + x1(1-x_2) + (1-x_1)x_2} + 2x_3 - 1 \end{aligned} \tag{15}$$

Substituting $x_3 = x_1 + x_2$, equation (15) implies

$$w_1^* + w_2^* = \frac{(-1 + x_1 + x_2)^2(x_1 + x_2)(-1 + 2x_1x_2)}{(-1 + x_1^2 + x_1x_2)(-1 + x_2^2 + x_1x_2)} .$$

This is easily proven to be ≤ 0, with equality when $x_1 + x_2 = x_3 = 1$ or $x_1 = x_2 = x_3 = 0$. Denote these special cases by (i) and (ii), respectively. Therefore, unless we are in cases (i) or (ii), $\bigtriangledown_{w_1,w_2} f = 0$ is infeasible in the interesting region.

We can say more than that: It can be easily verified that $w_3^* \geq 0$, therefore, $w_1^* + w_2^* < w_3^*$ (unless we are in cases (i) or(ii)). Let $\mathbf{w}^* = (w_1^*, w_2^*, w_3^*)$. Let H be the hyperplane $\{w_3 = w_1 + w_2\}$. By our last claim, $\mathbf{w}^*$ and P_w are on either side of H (unless we are in cases (i) or (ii)). Let H^+ denote the closed halfspace determined by H that intersects P_w. Unless at least two x_i's equal 1 (in our case this means $(x_1, x_2, x_3) = (1, 0, 1)$ or $(0, 1, 1)$ - call this case (iii)), it is easy to see that $f \to -\infty$ as $\|\mathbf{w}\| \to \infty$. This is because the coefficient of w_i^2 in f are strictly negative. Therefore, the maximum of f on $\mathbf{w} \in H^+$ (remember: $\mathbf{x}$ is fixed) is obtained on H (remember: (15) has only one solution). We postpone cases (i),(ii) and (iii) to the end of the proof. So we need only consider f on

$$(P_x \cap \{x_3 = x_1 + x_2\}) \times (P_w \cap \{w_3 = w_1 + w_2\}.$$

From now on, we assume both $x_3 = x_1 + x_2$ and $w_3 = w_1 + w_2$. Let $h(x_1, x_3, w_1, w_3) = f(x_1, x_3 - x_1, x_3, w_1, w_3 - w_1, w_3)$. We investigate the 3 cases: $w_1 = 0$, $w_1 = w_3$ and $\partial h/\partial w_1 = 0$.

1. If $w_1 = 0$, then we define $y(x_1, x_3, w_3) = h(x_1, x_3, 0, w_3)$. We want to show that $y \leq 0$ in the interesting region $0 \leq x_1 \leq x_3, 0 \leq w_3 \leq 1$. If $w_3 = 0$, then we obtain the following polynomial in x_1, x_3 of total degree 3 which is ≤ 0 in the interesting region (proof omitted):

$$\frac{4}{3}x_1^2 - \frac{4}{3}x_1x_3 - 4x_1^2x_3 - \frac{4}{3}x_3^2 + 4x_1x_3^2 .$$

If $w_3 = 1$, then we obtain the following polynomial in x_1, x_3 of total degree 3 which is ≤ 0 (proof omitted):

$$-\frac{4}{3} - 4x_1 - 2x_1^2 + \frac{8}{3}x_3 + \frac{16}{3}x_1x_3 + \frac{4}{3}x_1^2x_3 - \frac{4}{3}x_3^2 - \frac{4}{3}x_1x_3^2 .$$

Requiring $\partial y/\partial w_3 = 0$ implies

$$w_3 = \gamma(x_1, x_3) = \frac{-3x_1 - 2x_1^2 + 2x_3 + 4x_1x_3 + 4x_1^2x_3 - 4x_1x_3^2}{2 + x_1^2 - 2x_1x_3} .$$

Let $z(x_1,x_3) = (2+x_1^2-2x_1x_3)y(x_1,x_3,\gamma(x_1,x_3))$. Since $2+x_1^2-2x_1x_3 \geq 0$, we need to prove that $z(x_1,x_3) \leq 0$ in the region $0 \leq x_1 \leq x_3 \leq 1$. Calculations show that

$$z(x_1,x_3) = \frac{2}{3}x_1(13x_1 + 12x_1^2 + 6x_1^3 - 16x_3 - 44x_1x_3 - 46x_1^2x_3 - 22x_1^3x_3 \\ +28x_3^2 + 58x_1x_3^2 + 66x_1^2x_3^2 + 16x_1^3x_3^2 - 12x_3^3 - 44x_1x_3^3 - 32x_1^2x_3^3 + 16x_1x_3^4)\ .$$

We want to show that $z \leq 0$ in the interesting region $0 \leq x_1 \leq x_3 \leq 1$. For the sake of simplicity, redefine z by ignoring the term $\frac{2}{3}x_1$. We first prove our claim for $x_1 = 0$: In this case, $z = -16x_3 + 28x_3^2 - 12x_3^3$. It is elementary to check that this expression is ≤ 0 when $x_3 \in [0,1]$. Assume now that $x_1 > 0$. Define the set $J = (0, 0.55] \cup [0.7, 1]$. We define 3 sets as follows:

$$\begin{aligned} l_1 &= \{(x_1,x_3)|x_3 = x_1,\ x_1 \in (0,1]\} \\ l_2 &= \{(x_1,x_3)|x_3 = 1,\ x_1 \in (0,1]\} \\ l_3 &= \{(x_1,x_3)|x_3 = 1 + x_1/2,\ x_1 \in (0,1]\} \\ l_4 &= \{(x_1,x_3)|x_3 = 1.8,\ x_1 \in J\}\ . \end{aligned}$$

On l_1, it is elementary to verify that $z = -x_1 - 4x_1^2 + 6x_1^3$, which is negative when $x_1 \in (0,1]$. On l_2 it is elementary to verify that $z = -x_1$, which is negative when $x_1 \in (0,1]$. On l_3 it is elementary to verify that $z = x_1 - x_1^3 + x_1^5$, which is obviously positive when $x_1 \in (0,1]$. On l_4 it is elementary to verify that $z = (-5040 + 20671x_1 - 27240x_1^2 + 11400x_1^3)/625 < 0$ when $x_1 \in J$. It is now clear that $z < 0$ in the interesting region restricted to $\{x_1 \in J\}$: Fix $x_1 \in J$, and notice that $z \to +\infty$ when $x_3 \to \pm\infty$. By counting the number of sign changes as a function of x_3 we conclude that z must be negative in the interesting region restricted to $\{x_1 \in J\}$. It remains to analyze z in the region

$$R = \{(x_1,x_3)|x_1 \in K, x_1 \leq x_3 \leq 1\}\ ,$$

where K is the open interval $(0.55, 0.7)$. We first show that in the interior of R, the equation $\partial z/\partial x_3 = 0$ has a solution only for $x_3 < 0.7$. We prove this by arguing (details omitted) that:

$$\begin{aligned} &\partial z/\partial x_3 < 0 \text{ when } x_3 = 1.7,\ x_1 \in K \\ &\partial z/\partial x_3 > 0 \text{ when } x_3 = 1,\ x_1 \in K \\ &\partial z/\partial x_3 > 0 \text{ when } x_3 = 0.7,\ x_1 \in K\ . \end{aligned}$$

Then by a simple counting of sign changes of $\partial z/\partial x_3$ as a function of x_3 (which is a degree 3 polynomial when $x_1 \in K$ fixed) we reach the required conclusion. It remains to analyze z in the region

$$Q = \{(x_1,x_3)|x_1,x_3 \in K, x_1 \leq x_3\}\ .$$

We show that z does not have any local maxima in the interior of Q. It suffices to show that $\partial^2 z/\partial x_3^2 > 0$ in the interior of Q. We prove this by arguing (details omitted) that:

$$\begin{aligned} &\partial^2 z/\partial x_3^2 < 0 \text{ when } x_3 = 1,\ x_1 \in K \\ &\partial^2 z/\partial x_3^2 > 0 \text{ when } x_3 = 0.7, x_1 \in K \\ &\partial^2 z/\partial x_3^2 > 0 \text{ when } x_3 = x_1, x_1 \in K\ , \end{aligned}$$

Then by a simple counting of sign changes of $\partial^2 z/\partial x_3^2$ as a function of x_3 (which is a degree 2 polynomial when $x_1 \in K$ fixed) we reach the required conclusion.

2. Assume $w_1 = w_3$: If $w_1 = w_3 = 0$, then

$$h = 4x_1^2/3 - 4x_1x_3/3 - 4x_1^2x_3 - 4x_3^2/3 + 4x_1x_3^2 .$$

The can be shown to be ≤ 0 in the interesting region (proof omitted). If $w_1 = w_3 = 1$ then

$$h = \frac{2}{3}(-2 + 6x_1 - 3x_1^2 - 2x_3 - 2x_1x_3 + 2x_1^2x_3 + 3x_3^2 - 2x_1x_3^2 .$$

This polynomial can be shown to be ≤ 0 in the interesting region (proof omitted). Setting $p(x_1, x_3, w_1) = h(x_1, x_3, w_1, w_1)$, and forcing $\partial p/\partial w_1 = 0$ implies

$$w_1 = \delta(x_1, x_2) = \frac{3x_1 - 2x_1^2 - x_3 + 4x_1^2x_3 + 2x_3^2 - 4x_1x_3^2}{2 + x_1^2 - x_3^2} .$$

Let $q(x_1, x_3) = (2 + x_1^2 - x_3^2)p(x_1, x_3, \delta(x_1, x_3))$. We want to prove that $q \leq 0$ in $0 \leq x_1 \leq x_3 \leq 1$. Calculations show that

$$q(x_1, x_3) = \frac{2}{3}(x_1 - x_3)(13x_1 - 12x_1^2 + 6x_1^3 + 3x_3 - 20x_1x_3 + 28x_1^2x_3 - 22x_1^3x_3 + 4x_3^2 - 16x_1x_3^2 + 16x_1^3x_3^2 - 6x_3^3 + 22x_1x_3^3 - 16x_1^2x_3^3) .$$

To prove this, we omit the $\frac{2}{3}(x_1 - x_3)$ factor and redefine

$$q(x_1, x_3) = (13x_1 - 12x_1^2 + 6x_1^3 + 3x_3 - 20x_1x_3 + 28x_1^2x_3 - 22x_1^3x_3 + 4x_3^2 - 16x_1x_3^2 + 16x_1^3x_3^2 - 6x_3^3 + 22x_1x_3^3 - 16x_1^2x_3^3) .$$

We want to show that $q \geq 0$ in $0 \leq x_1 \leq x_3 \leq 1$. We omit the proof of this on the boundary of the region. We prove this for two subcases.

(a) Case $0.4 \leq x_1 < 1$. We define three line segments:

$$l_1 = \{(x_1, x_3) | x_3 = x_1, 0.4 \leq x_1 < 1\}$$
$$l_2 = \{(x_1, x_3) | x_3 = 1, 0.4 \leq x_1 < 1\}$$
$$l_3 = \{(x_1, x_3) | x_3 = 1.2, 0.4 \leq x_1 < 1\}$$

It is not hard to show that $q > 0$ on l_1 and l_2 and $q < 0$ on l_3. It is also not hard to see that $q \to \pm\infty$ as $x_3 \to \pm\infty$, when $0.4 \leq x_1 < 1$ (simply by showing that the coefficient of x_3^3 is positive for such x_1). Therefore, by counting sign changes, we conclude that $q \geq 0$ in the region.

(b) Case $0 < x_1 < 0.4$. We look for a point (x_1, x_3) in this region for which $\partial q/\partial x_1 = 0$.

$$\frac{\partial q}{\partial x_1} = 13 - 24x_1 + 18x_1^2 - 20x_3 + 56x_1x_3 - 66x_1^2x_3 - 16x_3^2 + 48x_1^2x_3^2 + 22x_3^3 - 32x_1x_3^3 .$$

We define four line segments

$$l_1 = \{(x_1, x_3) | x_3 = 0, 0 < x_1 < 0.4\}$$
$$l_2 = \{(x_1, x_3) | x_3 = 0.59, 0 < x_1 < 0.4\}$$
$$l_3 = \{(x_1, x_3) | x_3 = 0.65, 0 < x_1 < 0.4\}$$
$$l_4 = \{(x_1, x_3) | x_3 = 1, 0 < x_1 < 0.4\}$$

It is not hard to show that $\partial q/\partial x_1 > 0$ on l_1 and l_2, and that $\partial q/\partial x_1 < 0$ on l_3 and l_4. Also it is easy to see that $\partial q/\partial x_1 \to \pm\infty$ as $x_3 \to \pm\infty$ when $0 < x_1 < 0.4$. By counting sign changes, we conclude that $\partial q/\partial x_1$ can vanish in the interesting region only when $0.59 < x_3 < 0.65$. We are therefore left with the task of showing that $q \geq 0$ in the region

$$\{(x_1, x_3)|0.59 < x_3 < 0.65, 0 < x_1 < 0.4\} \; .$$

To do this we replace q with a function $\tilde{q}$ that lower bounds q, as follows: replace x_3 with 0.59 in any *positive* term of q, and with 0.65 in any *negative* term of q. The function $\tilde{q}$ thus obtained clearly lower bounds q, and it suffices to show that $\tilde{q} \geq 0$ in the region $0 < x_1 < 0.4$. It can be easily verified that

$$\tilde{q} = \frac{30293}{20000} - \frac{1120831}{500000}x_1 + \frac{63}{500}x_1^2 - \frac{3413}{1250}x_1^3 \; .$$

Proving that $\tilde{q} \geq 0$ in the region $0 < x_1 < 0.4$ is omitted.

3. Assume $\partial h/\partial w_1 = 0$. We distinguish between 3 subcases: case $x_1 = 0$, case $x_1 = x_3$ and case $\partial h/\partial x_1 = 0$.

   (a) $(\partial h/\partial w_1 = 0, x_1 = 0)$. In this case we get

   $$w_1 = \nu(x_3, w_3) = \frac{-w_3 + 3x_3 - 2x_3^2}{-2 + x_3^2} \; .$$

   Substituting, we define $v(x_3, w_3) = (2 - x_3^2)h(0, x_3, \nu(x_3, w_3), w_3)$. We need to show that $v \leq 0$ in $0 \leq \mathbf{x}_3 \leq 1, 0 \leq w_3 \leq 1$. We have an additional constraint, namely, the assertion $w_1 = \nu(x_3, w_3) \geq 0$, which implies $w_3 \geq 3x_3 - 2x_3^2$. We omit the technical proof of $v \leq 0$ in this region.

   (b) $(\partial h/\partial w_1 = 0, x_1 = x_3)$. In this case we get

   $$w_1 = \mu(x_3, w_3) = \frac{-w_3 - 3x_3 + 2x_3^2 + w_3 x_3^2}{-2 + x_3^2} \; .$$

   We define $u(x_3, w_3) = (2 - x_3^2)h(x_3, x_3, \mu(x_3, w_3), w_3)$. We want to prove $u \leq 0$ in the region $0 \leq x_3 \leq 1$, $0 \leq w_3 \leq 1$, with the additional constraint $w_1 = \mu(x_3, w_3) \leq w_3$, which implies $w_3 \geq 3x_3 - 2x_3^2$. We omit the proof of this assertion.

   (c) $(\partial h/\partial w_1 = 0, \partial h/\partial x_1 = 0)$. Calculations show that this implies $w_1 = w_3/2, x_1 = x_3/2$. Thus we define

   $$r(w_3, x_3) = h(x_3/2, x_3, w_3/2, w_3) = \frac{1}{3}(-3w_3^2 + 2w_3x_3 - 5x_3^2 + 6w_3x_3^2 + w_3^2x_3^2 + 3x_3^3 - 4w_3x_3^3) \; .$$

   We show that $r \leq 0$ in the region $0 \leq x_3 \leq 1, 0 \leq w_3 \leq 1$. If $w_3 = 0$, then it can be easily shown that $r = -5x_3^2/3 + x_3^3 \leq 0$. If $w_3 = 1$, then $r = (-3 + 2x_3 + 2x_3^2 - x_3^3)/3$, which is also ≤ 0 (proof omitted). If $\partial r/\partial w_3 = 0$, then

   $$w_3 = \rho(x_3) = \frac{-x_3 - 3x_3^2 + 2x_3^3}{-3 + x_3^2} \; .$$

   We define $s(x_3) = (3 - x_3^2)r(x_3, \rho(x_3))$. We need to prove that $s \leq 0$ when $0 \leq x_3 \leq 1$. Simple calculations show that

   $$s(x_3) = \frac{1}{3}(-2 + x_3)(-1 + x_3)x_3^2(1 + x_3)(-7 + 4x_3) \leq 0 \; .$$

We are left with cases (i), (ii) and (iii), defined above.

Case (i). $x_1 + x_2 = x_3 = 1$. Let $g(x_1, w_1, w_2, w_3) = f(x_1, 1 - x_1, 1, w_1, w_2, w_3)$. In this case, $\bigtriangledown_{w_1,w_2} g = 0$ exactly when $w_1 = -1 + 2x_1$ and $w_2 = 1 - 2x_1$. This implies that $w_1 + w_2 = 0$, meaning that $\mathbf{w}^*$ is again on the other side of H (w.r.t P_w), unless $w_3 = 0$. The proof therefore continues as before. If $w_1 = w_2 = w_3 = 0$, we get $g = 4(-1 + 2x_1(1 - x_1))/3 \leq 0$.

Case(ii). $x_1 = x_2 = x_3 = 0$. This implies $f = -2(w_1^2 + w_2^2 + w_3^2) \leq 0$.

Case(iii). Assume $x_1 = 1, x_2 = 0, x_3 = 1$. In this case we have,

$$f = -(4 + 2w_1(2 - w_1) + 2w_3(2 - w_3))/3 \leq 0 \ .$$

□

5 Proof of Theorem 5

Let piv, opt be defined as in Section 4. Let

$$f(\mathbf{x}, \mathbf{w}) = piv - \frac{5}{2} opt \ ,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. We prove that $f \leq 0$ on

$$\begin{aligned} 0 \leq w_i \leq 1 \quad i = 1, 2, 3 \\ 0 \leq x_i \leq 1 \quad i = 1, 2, 3 \\ x_1 \leq x_2 + x_3,\ x_2 \leq x_3 + x_1,\ x_3 \leq x_1 + x_2 \end{aligned} \tag{16}$$

Let $P_{\mathbf{x}}$ denote the polytope corresponding to $\mathbf{x}$ and P_w denote the polytope (cube) corresponding to $\mathbf{w}$.

Clearly f is linear in $\mathbf{w}$ when $\mathbf{x}$ is fixed. So it suffices to investigate f on the vertices of $P_{\mathbf{w}}$. Due to the symmetric structure of f, it will suffice to investigate f on the points $\mathbf{w} = (0, 0, 0)$, $\mathbf{w} = (0, 0, 1)$, $\mathbf{w} = (1, 1, 0)$, $\mathbf{w} = (1, 1, 1)$.

1. Case $\mathbf{w} = (0, 0, 0)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 0, 0, 0)$. So

$$g = -\frac{1}{2}(x_1 + x_2 + x_3) - 2(x_1x_2 + x_2x_3 + x_3x_1) + \frac{15}{2} x_1x_2x_3 \ .$$

Since g is linear in each one of x_i when the others are fixed, we can assume that either $x_3 = x_1 + x_2$, $x_2 = x_1 + x_3$, $x_1 = x_2 + x_3$ or $(x_1, x_2, x_3) = (1, 1, 1)$ (see Section 4). The latter case will be taken care of at the end of the proof. Due to symmetry, we can consider only $x_3 = x_1 + x_3$.

- Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \leq 0$ on $0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1$.

$$h(x_1, x_2) = -(x_1 + x_2) - 2(x_1^2 + x_2^2) + \frac{15}{2}(x_1x_2^2 + x_1^2x_2) \ .$$

If $x_1 = 0$ then $h = -x_2(1 + 2x_2) \leq 0$. Similarly if $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = (-6 + 11x_2 - 11x_2^2)/2 \leq 0$. There is no solution to $\bigtriangledown_{x_1,x_2} h = 0$ in the region of interest.

2. Case $\mathbf{w} = (0,0,1)$. Let $g(\mathbf{x}) = f(\mathbf{x},0,0,1)$. So

$$g = -\frac{3}{2} - \frac{5}{2}(x_1 + x_2) + \frac{9}{2}x_3 - 2(x_1x_3 + x_2x_3) + \frac{7}{2}x_1x_2 + \frac{5}{2}x_1x_2x_3 \ .$$

Due to the structure of f and the symmetries, it suffices to investigate g when either $x_1 = x_2 + x_3$, $x_3 = x_1 + x_2$ or $\mathbf{x} = (1,1,1)$. The latter case will be taken care of separately.

(a) Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \leq 0$ on $0 \leq x_1$, $0 \leq x_2$, $x_1 + x_2 \leq 1$.

$$h(x_1, x_2) = -\frac{3}{2} + 2(x_1 + x_2) - 2(x_1^2 + x_2^2) + \frac{5}{2}(x_1x_2^2 + x_1^2x_2) - \frac{1}{2}x_1x_2 \ .$$

If $x_1 = 0$ then $h = -\frac{3}{2} + 2(1 - x_2)x_2 \leq 0$ on $x_2 \in [0,1]$. Similarly holds when $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -\frac{3}{2}(1 - 2x_2)^2 \leq 0$. We completed checking the boundary. There is no solution to $\bigtriangledown_{x_1,x_2} h = 0$ in the interesting region.

(b) Subcase $x_1 = x_2 + x_3$. Let $h(x_2, x_3) = g(x_2 + x_3, x_2, x_3)$. We want to show that $h \leq 0$ on $0 \leq x_2$, $0 \leq x_3$, $x_2 + x_3 \leq 1$.

$$h(x_1, x_2) = -\frac{3}{2} + -5x_2 + 2x_3 + \frac{7}{2}x_2^2 - 2x_3^2 + \frac{5}{2}(x_2^2x_3 + x_2x_3^2) - \frac{1}{2}x_2x_3 \ .$$

If $x_2 = 0$ then $h = -3/2 - 2(-1 + x_3)x_3 \leq 0$. If $x_3 = 0$ then $h = -\frac{3}{2} + x_2(-5 + \frac{7}{2}x_2) \leq 0$ when $x_2 \in [0,1]$. If $x_2 = 1 - x_3$ then $h = (-6 + 4x_3 - x_3^2)/2 \leq 0$ on $x_2 \in [0,1]$. There is no solution to $\bigtriangledown_{x_2,x_3} h = 0$ in the interesting region.

3. Case $\mathbf{w} = (1,1,0)$. Let $g(\mathbf{x}) = f(\mathbf{x},1,1,0)$. So

$$g = -3 + \frac{5}{2}(x_1 + x_2) - \frac{9}{2}x_3 + \frac{7}{2}(x_1x_3 + x_2x_3) - 2x_1x_2 - \frac{5}{2}x_1x_2x_3 \ .$$

Due to the structure of f and the symmetries, it suffices to investigate g when either $x_1 = x_2 + x_3$,$x_3 = x_1 + x_2$, or $\mathbf{x} = (1,1,1)$. The latter case will be taken care of separately.

(a) Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \leq 0$ on $0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1$.

$$h(x_1, x_2) = -3 - 2(x_1 + x_2) + \frac{7}{2}(x_1^2 + x_2^2) - \frac{5}{2}(x_1x_2^2 + x_1^2x_2) + 5x_1x_2 \ .$$

If $x_1 = 0$ then $h = -3 + x_2(-2 + \frac{7}{2}x_2) \leq 0$ when $x_1 \in [0,1]$. Similarly holds when $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = 3(-1 - 3x_2 + 3x_2^2)/2 \leq 0$ when $x_2 \in [0,1]$. We completed checking the boundary. The only solution to $\bigtriangledown_{x_1,x_2} h = 0$ in the interesting region is $x_1 = x_2 = 2(6 - \sqrt{21})/15$, at which $h = (-243 - 112\sqrt{21})/225 \leq 0$.

(b) Subcase $x_1 = x_2 + x_3$. Let $h(x_2, x_3) = g(x_2 + x_3, x_2, x_3)$. We want to show that $h \leq 0$ on $0 \leq x_2, 0 \leq x_3, x_2 + x_3 \leq 1$.

$$h(x_2, x_3) = -3 + 5x_2 - 2x_3 - 2x_2^2 + \frac{7}{2}x_3^2 - \frac{5}{2}(x_2^2x_3 + x_2x_3^2) + 5x_2x_3 \ .$$

If $x_2 = 0$ then $h = -3 + x_3(-2 + \frac{7}{2}x_3) \leq 0$ when $x_3 \in [0,1]$. If $x_3 = 0$ then $h = -3 + x_2(5 - 2x_2) \leq 0$ for $x_2 \in [0,1]$. If $x_2 = 1 - x_3$ then $h = -x_3/2 - x_3^2 \leq 0$ on $x_3 \in [0,1]$. There is no solution to $\bigtriangledown_{x_2,x_3} h = 0$ in the interesting region.

4. Case $\mathbf{w} = (1,1,1)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 1, 1, 1)$. So

$$g = -\frac{9}{2} + \frac{1}{2}(x_1 + x_2 + x_3) + \frac{7}{2}(x_1x_2 + x_2x_3 + x_3x_1) - \frac{15}{2}x_1x_2x_3 \ .$$

   Since g is linear in each one of x_i when the others are fixed, we can assume that either $x_3 = x_1+x_2$, $x_2 = x_1+x_3$,$x_1 = x_2+x_3$ or $(x_1, x_2, x_3) = (1, 1, 1)$. The latter case will be taken care of separately. Due to symmetry, we can consider only $x_3 = x_1 + x_3$.

   - Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \leq 0$ on $0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1$.

$$h(x_1, x_2) = -\frac{9}{2} + (x_1 + x_2) + \frac{7}{2}(x_1^2 + x_2^2) - \frac{15}{2}(x_1x_2^2 + x_1^2x_2) + \frac{21}{2}x_1x_2 \ .$$

     If $x_1 = 0$ then $h = -\frac{9}{2} + x_2(1 + \frac{7}{2}x_2) \leq 0$ when $x_2 \in [0, 1]$. Similarly if $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = 4x_2(-1 + x_2)$, which is ≤ 0 when $x_2 \in [0, 1]$. There is no solution to $\bigtriangledown_{x_1,x_2} h$ in the interesting region.

It remaines to investigate f when $\mathbf{x} = (1, 1, 1)$. It is not hard to see that $f(1, 1, 1, \mathbf{w}) = 0$, as required. □

6 Proof of Theorem 6

Let piv, opt be defined as in Section 4. Let

$$f(\mathbf{x}, \mathbf{w}) = piv - 2opt \ ,$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. We prove that $f \leq 0$ on

$$\begin{aligned} 0 \leq w_i \leq 1 \quad i = 1, 2, 3 \\ 0 \leq x_i \leq 1 \quad i = 1, 2, 3 \\ w_1 \leq w_2 + w_3,\ w_2 \leq w_3 + w_1,\ w_3 \leq w_1 + w_2 \\ x_1 \leq x_2 + x_3,\ x_2 \leq x_3 + x_1,\ x_3 \leq x_1 + x_2 \end{aligned} \tag{17}$$

Let $P_{\mathbf{x}}$ denote the polytope corresponding to $\mathbf{x}$ and P_w denote the polytope (cube) corresponding to $\mathbf{w}$.

Clearly f is linear in $\mathbf{w}$ when $\mathbf{x}$ is fixed. So it suffices to investigate f on the vertices of $P_{\mathbf{w}}$. Due to the symmetric structure of f, it will suffice to investigate f on the points $\mathbf{w} = (0, 0, 0)$, $\mathbf{w} = (1, 1, 0)$, $\mathbf{w} = (1, 1, 1)$.

1. Case $\mathbf{w} = (0, 0, 0)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 0, 0, 0)$. So

$$g = -2(x_1x_2 + x_2x_3 + x_3x_1) + 6x_1x_2x_3 \ .$$

   Since g is linear in each one of x_i when the others are fixed, we can assume that either $x_3 = x_1+x_2$, $x_2 = x_1 + x_3$, $x_1 = x_2 + x_3$ or $(x_1, x_2, x_3) = (1, 1, 1)$. The latter case will be taken care of at the end of the proof. Due to symmetry, we can consider only $x_3 = x_1 + x_3$.

- Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \le 0$ on $0 \le x_1$, $0 \le x_2$, $x_1 + x_2 \le 1$.

$$h(x_1, x_2) = -2(x_1^2 + x_2^2) + 6(x_1 x_2^2 + x_1^2 x_2) - 6x_1 x_2 \ .$$

If $x_1 = 0$ then $h = -2x_2^2 \le 0$. Similarly if $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -2 + 4x_2(1 - x_2) \le 0$. The equation $\bigtriangledown_{x_1, x_2} h = 0$ has no solution in the interior of the interesting region.

2. Case $\mathbf{w} = (1, 1, 0)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 1, 1, 0)$. So

$$g = -2 + 2(x_1 + x_2) - 4x_3 + 3(x_3 x_1 + x_2 x_3) - 2x_1 x_2 - 2x_1 x_2 x_3 \ .$$

Due to the structure of f and the symmetries, it suffices to investigate g when either $x_1 = x_2 + x_3$, $x_3 = x_1 + x_2$, or $\mathbf{x} = (1, 1, 1)$. The latter case will be taken care of separately.

(a) Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \le 0$ on $0 \le x_1$, $0 \le x_2$, $x_1 + x_2 \le 1$.

$$h(x_1, x_2) = -2 - 2(x_1 + x_2) + 3(x_1^2 + x_2^2) - 2(x_1 x_2^2 + x_1^2 x_2) + 4x_1 x_2.$$

If $x_1 = 0$ then $h = -2 + x_2(-2 + 3x_2) \le 0$ on $x_2 \in [0, 1]$. Similarly if $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -1 + 4x_2(-1 + x_2) \le 0$ for $x_2 \in [0, 1]$. We completed checking the boundary. The only solution to $\bigtriangledown_{x_1, x_2} h = 0$ in the interesting region is $x_1 = x_2 = (5 - \sqrt{13})/6$, at which $h = (-19 - 13\sqrt{13})/27 \le 0$.

(b) Subcase $x_1 = x_2 + x_3$. Let $h(x_2, x_3) = g(x_2 + x_3, x_2, x_3)$. We want to show that $h \le 0$ on $0 \le x_2$, $0 \le x_3$, $x_2 + x_3 \le 1$.

$$h(x_2, x_3) = -2 + 4x_2 - 2x_3 - 2x_2^2 + 3x_3^2 - 2(x_2^2 x_3 + x_2 x_3^2) + 4x_2 x_3 \ .$$

If $x_2 = 0$ then $h = -2 + x_3(-2 + 3x_3) \le 0$ on $x_3 \in [0, 1]$. If $x_3 = 0$ then $h = -2 + x_2(4 - 2x_2) \le 0$. If $x_2 = 1 - x_3$ then $h = -x_3^2 \le 0$. There is no solution to $\bigtriangledown_{x_2, x_3} h = 0$ in the interior of the interesting region.

3. Case $\mathbf{w} = (1, 1, 1)$. Let $g(\mathbf{x}) = f(\mathbf{x}, 1, 1, 1)$. So

$$g = 3 + 3(x_1 x_2 + x_2 x_3 + x_3 x_1) - 6x_1 x_2 x_3 \ .$$

Since g is linear in each one of x_i when the others are fixed, we can assume that either $x_3 = x_1 + x_2$, $x_2 = x_1 + x_3$, $x_1 = x_2 + x_3$ or $(x_1, x_2, x_3) = (1, 1, 1)$. The latter case will be taken care of separately. Due to symmetry, we can consider only $x_3 = x_1 + x_3$.

- Subcase $x_3 = x_1 + x_2$. Let $h(x_1, x_2) = g(x_1, x_2, x_1 + x_2)$. We want to show that $h \le 0$ on $0 \le x_1$, $0 \le x_2$, $x_1 + x_2 \le 1$.

$$h(x_1, x_2) = -3 + 3(x_1^2 + x_2^2) - 6(x_1 x_2^2 + x_1^2 x_2) + 9x_1 x_2 \ .$$

If $x_1 = 0$ then $h = -3 + 3x_2^2 \le 0$ for $x_2 \in [0, 1]$. Similarly for $x_2 = 0$. If $x_1 = 1 - x_2$ then $h = -3x_2 + 3x_2^2 \le 0$ for $x_2 \in [0, 1]$. There is no solution to $\bigtriangledown_{x_1, x_2} h = 0$ in the interior of the interesting region.

It remaines to investigate $f(1, 1, 1, \mathbf{w})$. It is not hard to see that $f(1, 1, 1, \mathbf{w}) = 0$, as required.

□

References

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