Probabilistic Spaces of Boolean Functions of a Given Complexity: Generalities and Random k-SAT Coefficients

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In this paper we develop an approach to studying probabilistic spaces of boolean functions, namely recovering exact formulas for the event probabilities in terms of the moments. While this involves analyzing a large number of moments, there are situations in which this seems feasible to do; for the m-fold AND of a probability space of functions, there is a formula involving coefficients with a geometric intepretation (and which is otherwise quite simple). We investigate the coefficients involved in the k-SAT problem, where we give a formula for the 1-SAT coefficients and are able to understand a few of the 2-SAT coefficients.

1 Introduction

In this paper we begin an investigation of a set of problems via the following philosophy. Let $\mathcal{F}$ be a probability space of Boolean valued (i.e. $\mathbf{B} = \{0, 1\}$ valued) functions on a fixed set S; i.e. the events of the space are functions $f: S \to \{0, 1\}$. Viewing the elements $s \in S$ as random variables, we can calculate the moments of the probability space with respect to the elements of S. In fact, because the random variables are $\{0, 1\}$ valued, $s^n = s$ for all $n \geq 1$ and $s \in S$, and we can restrict our attention to moments of the form

$$E(T) \stackrel{\text{def}}{=} \mathrm{E}\left\{\prod_{t \in T} t\right\},$$

for subets $T \subset S$. Once we understand these 2^s moments, $s = |S|$, in principle we can recover the probability of every event and reconstruct the space.

In complexity theory one typically has a sequence of sets of Boolean functions on n variables, $\mathcal{F}_1, \mathcal{F}_2, \ldots$, indexed by their complexity, such that $\mathcal{F}_m$ is derived from previous $\mathcal{F}_i$'s in a simple way. For example, $\mathcal{F}_m$ might consist of all m-fold AND's of $\mathcal{F}_1$, i.e. AND's

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of m elements of $\mathcal{F}_1$, or might consist of all depth m NAND tree's with $\mathcal{F}_i$ leaves, i.e. $\mathcal{F}_m$ consisting of all NAND's of two elements of $\mathcal{F}_{m-1}$. In such a situation, if we impose a probability distribution on $\mathcal{F}_1$ and impose a probablistic form of the deriving rule, then the $\mathcal{F}_m$ become probability spaces. One can attempt to study such spaces by the method suggested above. One might express pessimism at the fact that 2^{2^n} moments are involved in the method and must somehow be understood. Nonetheless, we will show in this paper that this method can be used to study such problems and, at least in the case of m-fold AND, yields interesting connections with geometry. It is the hope that these methods, when better understood, might yield lower and upper bounds in complexity theory, i.e. that one might be able to understand the moments well enough to tell whether or not certain event probabilities vansish. Based on the connections with geometry we also suggest a possible homological approach to lower bounds, which may not require analyzing the entire event probability recovering formula.

In the case of the m-fold AND of a fixed space, $\mathcal{F}_1$, recovering event probabilities yields simple summations involving certain coefficients which are independent of m (depending only on $\mathcal{F}_1$). These coefficients have a geometric interpretation, essentially the Euler characteristic of an abstract simplicial complex. We study these coefficients and complexes in the example of the "random k-SAT" problem. Here the author only has an explicit formula for the coefficients for the case $k = 1$ (where much about the original problem is easy to understand). For $k = 2$ the author has computed these coefficients for small values of n; these calculations suggest a simple formula for the first of these coefficients, which we prove true in general; we also calculate general formulas for the second and third coefficients, which turn out to be fairly simple. We make some further progress towards understanding these coefficients, but at present we do not understand them well enough to estimate the summation involving them which gives the event probabilities. The proofs we give are based on a general fact about Euler characteristics of simplicial complexes, proposition 5.1, which is essentially a special case of the Čech cohomology computation; the idea to express the Euler characteristic in terms of the "overcover" was suggested to the author by Michael Ben-Or, for which the author is indebted to him.

To better understand the coefficients and sums involving them, in the possible absence of simple formulas for the coefficients (for $k \geq 2$), we study the geometry of the associated simplicial complexes. We consider their reduced[1] betti numbers, since the reduced Euler characteristic is just the alternating sum of the reduced betti numbers. These complexes often, but not always, have the property that all but one of the reduced betti numbers vanish, i.e. real reduced homology occurs only in one dimension. We prove that this is always the case when $k = 1$ (more strongly, we show this for homology over the integers). While this phenomenon of the purity of the reduced homology is fairly common in combinatorics (see, for example [Bjö84]), the methods used there do not directly apply here— in fact our simplicial complexes do not even have their maximal faces all of one dimension. Yet, at least for $k = 1$, there is a purity phenomenon, the Euler characteristic coming entirely from this one dimension; for $k = 2$ there exist counterexamples, but the smaller simplicial complexes, in particular that responsible for the first coefficient described in the previous paragraph, has

[1] The reduced betti numbers are the same as the betti numbers except that the 0-th reduced betti number is one less than the 0-th betti number. Reduced homology, obtained by augmenting the standard chain complex by $\mathbf{Z}$, and the resulting reduced betti numbers, are much more convenient for the present discussion.

purity for $n = 2, 3, 4$. We believe that further understanding the geometry involved could yield information about the resulting coefficients, and might give insights into a possible approach to lower bounds based on homology, which we explain in the conculsion.

In order to perform computer-aided calculations of the betti numbers, we develop some techniques which seem to work well in practice. They are based on the Laplacian of the chain complex. They yield an especially quick way of checking whether or not a betti number vanishes, or of computing a betti number of small size (relative to the size of the chain complexes).

In §2 we discuss the general method and the geometric interpretation of m-fold AND coefficients. In §3 we recall the k-SAT problem, give a formula for 1-SAT coefficients, and give and remark on a table of 2-SAT coefficients for small values of n. In §4 we prove the purity for the 1-SAT complexes and give the results of computations showing that purity does not hold in general. In §5 we give some general techniques for computing Euler characteristics to be used later. In §6 we prove the simple formula for the first 2-SAT coefficient, and derive a formula for the second coefficient. In §7 we derive a formula for the third coefficient, and prove that in general the i-th 2-SAT coefficient is given by a system of linear recurrences whose coefficients are polynomials in n. In §8 we describe the method used for homology calculations

We conclude the paper with some remarks about further directions of study of this method. Many upper bounds, i.e. constructions, in complexity theory and combinatorics are based on counting arguments, which is essentially a "first-moment" analysis of a probability space. Here we suggest that it is not impossible to perform a complete analysis of the moments, at least in certain cases. We hope that further study of these methods and the connection with geometry may yield results not provable by presently known techniques.

The author learned of the k-SAT problem from Miklös Santha and Jacques Stern. Michael Ben-Or made a suggestion on the calculation of the coefficients, which lead to proposition 5.1 and the proof of equation 3.3 (and, in general, sections 6 and 7). The author wishes to thank them for discussions and comments, as well as Lynne Bulter, Vasek Chvátal, Bruce Donald, Michael Hirsch, Richard Stanley, Fernandez de la Vega, and Avi Wigderson.

2 Generalities

Let us return to the general situation of a probability space, $\mathcal{F}$, of Boolean valued functions on a fixed set S. Since the expected value of a $\{0,1\}$ valued random variable is just the probability that it equals 1, we have

$$E_{\mathcal{F}}(T) \stackrel{\text{def}}{=} \mathrm{E}_{\mathcal{F}}\left\{\prod_{t\in T} t\right\} = \mathrm{Pr}_{f\in\mathcal{F}}\{f|_T = 1\},$$

where $f|_T$ denotes f's restriction to T. Similarly, for disjoint subsets, A, B of S we have the probability that an element of $\mathcal{F}$ is 0 on A and 1 on B is

$$\mathrm{Pr}_{f\in\mathcal{F}}\{f|_A = 0, \text{ and } f|_B = 1\} = \mathrm{Pr}_{f\in\mathcal{F}}\left\{\left(\prod_{s\in A}(1-s)\prod_{s\in B} s\right) = 1\right\}$$

$$= \mathrm{E}\left\{\prod_{s\in A}(1-s)\prod_{s\in B} s\right\} = \sum_{B\subset C\subset A\cup B} (-1)^{|C|} E(C).$$

In particular, for a Boolean function g on S we have

$$\Pr_{f\in\mathcal{F}}\{f=g\} = \sum_{O\subset C\subset S} (-1)^{|C|}E(C),$$

where O is the subset of S where g takes the value 1, and in particular,

$$\Pr_{f\in\mathcal{F}}\{f=0\} = \sum_{C\subset S} (-1)^{|C|}E(C). \tag{2.1}$$

The above formulas enable us to recover the event probabilities from the moments of the space. They are restatements (or a concise proof) of the inclusion-exclusion principle.

Given probability spaces $\mathcal{F}_1,\ldots,\mathcal{F}_m$ of Boolean functions on S, we define probability spaces $\mathrm{NAND}(\mathcal{F}_1,\mathcal{F}_2)$, $\mathrm{AND}(\mathcal{F}_1,\ldots,\mathcal{F}_m)$, $\neg\mathcal{F}_1$, etc. in the obvious way (e.g. $\mathrm{NAND}(\mathcal{F}_1,\mathcal{F}_2)$ is obtained by independently choosing f_i in $\mathcal{F}_i$, $i=1,2$, and taking their NAND). By $\mathrm{AND}_m(\mathcal{F})$ we mean the space obtained by independently choosing $f_1,\ldots,f_m$ in $\mathcal{F}$ and taking their AND. It is easy to evaluate the $E(T)$'s of these derived spaces in terms of the old spaces.

Proposition 2.1

$$E_{\neg\mathcal{F}}(T) = \sum_{R\subset T}(-1)^{|R|}E_{\mathcal{F}}(T), \qquad E_{\mathrm{AND}(\mathcal{F}_1,\ldots,\mathcal{F}_m)}(T) = E_{\mathcal{F}_1}(T)\cdots E_{\mathcal{F}_m}(T),$$

$$E_{\mathrm{NAND}(\mathcal{F},\mathcal{G})}(T) = \sum_{R\subset T}(-1)^{|R|}E_{\mathcal{F}}(R)E_{\mathcal{G}}(R).$$

As an example of the above, given n let $\mathcal{F}_1$ be the space of Boolean functions on n variables with each of $\{x_1,\overline{x_1},\ldots,x_n,\overline{x_n}\}$ occuring with probability $1/2n$. Let $\mathcal{F}_i$ for $i>1$ be given by $\mathrm{NAND}(\mathcal{F}_{i-1},\mathcal{F}_{i-1})$. Given a Boolean function, f, let $i(f)$ denote the smallest i for which f occurs in $\mathcal{F}_i$ (with non-zero probability). It is easy to see that $i(f)$ measures the circuit depth complexity of f to within a constant factor. The author has tried to understand the moments and resulting formulas in this case, but has made very limited progress in doing so at present.

As another example, let $\mathcal{F}_1$ be a probability space of Boolean functions on S consisting of N functions, $f_1,\ldots,f_N$ each occuring with probability $1/N$, and let Z_i,O_i denote the sets where f_i is $0,1$ respectively. Let $\mathcal{F}_m$ be given as $\mathrm{AND}_m(\mathcal{F}_1)$. By equation 2.1 and proposition 2.1 we have

$$\Pr_{f\in\mathcal{F}_m}\{f=0\} = \sum_{i=0}^{N} c_i\left(\frac{i}{N}\right)^m, \tag{2.2}$$

where the c_i are independent of m and are given by

$$c_i = \sum_{R \text{ contains exactly } i \text{ of the } O_j\text{'s}} (-1)^{|R|}.$$

These c_i can be given a geometric interpretation. Namely, let

$$\mathcal{A}_i = \{T\subset S \mid T \text{ contains} \le i \text{ of the } Z_j\text{'s}\} = \{T\subset S\mid \overline{T} \text{ is a subset of} \le i \text{ of the } O_j\text{'s}\}.$$

Then $\mathcal{A}_i$ is an abstract simplicial complex, i.e. is closed under taking subsets, and we have

$$\left(\sum_{R \text{ contains } \leq i \text{ of the } O_j\text{'s}} (-1)^{|R|}\right)(-1)^{|S|} = \sum_{T \in \mathcal{A}_i} (-1)^{|T|} = -\tilde{\chi}(\mathcal{A}_i) = \sum_{j=0}^{N-1} \tilde{h}_j(\mathcal{A}_i),$$

where $\hat{\chi}$ is the reduced Euler characteristic, and $\tilde{h}_j$ are the reduced betti numbers (see [Mun84]). In summary we have:

Proposition 2.2

$$\Pr_{f \in \mathcal{F}_m}\{f = 0\} = \sum_{i=0}^{N} c_i \left(\frac{i}{N}\right)^m,$$

where the c_i are given by the reduced Euler characteristics

$$(-1)^{|S|} c_i = \tilde{\chi}(\mathcal{A}_{i-1}) - \tilde{\chi}(\mathcal{A}_i)$$

of the associated abstract simplicial complexes $\mathcal{A}_i$.

3 Random k-SAT

We now study a specific example, namely that of the "random k-SAT" problem; we begin by describing random 3-SAT. For a given n we consider the family $\mathcal{F}_m$ given as $\text{AND}_m(\mathcal{F}_1)$, where $\mathcal{F}_1$ consists of the $8\binom{n}{3}$ non-trivial (i.e. $\not\equiv 1$) disjunctions of the variables $x_1, \ldots, x_n$ and their negations. In other words, $\mathcal{F}_m$ is a randomly chosen 3-CNF formula with m clauses. The random 3-SAT problem asks for what values of m is this formula likely to be satisfiable. For example, the standard counting argument shows that

$$\Pr_{f \in \mathcal{F}_m}\{f = 0\} \geq 1 - 2^n (7/8)^m,$$

and so for any $\epsilon > 0$ we have that for $m \geq (\epsilon + \log_{8/7} 2)n$ the formula is exponentially unlikely to be satisfiable. One can also show that there exists a constant $c > 0$ such that for $m \leq cn$ a random 3-CNF is likely to be satisfiable. One can ask if there exists a threshold ratio of m/n, i.e. a constant α such that for any ϵ we have that for $m \leq (\alpha - \epsilon)n$ and $\geq (\alpha + \epsilon)n$ it is respectively likely and unlikely that a random 3-CNF of length m is satisfiable; if so, one can ask how $\Pr_{f \in \mathcal{F}_m}\{f = 0\}$ for m near αn.

Originally, the random 3-SAT arose as a model for testing heuristics used to solve 3-CNF's (see [CF90] and the references there). Proving that an algorithm almost always returns a satisfying assignment for m, n in a certain range shows, in particular, that random 3-CNF's in this range are almost always satisfiable.

In general, one can form the random k-SAT problem for any fixed k and ask similar questions. For $k = 1$ it is clear from the "birthday paradox" that for $m \ll \sqrt{n}$ and $m \gg \sqrt{n}$ the random 1-SAT will respectively satisfiable and unsatisfiable with high probability; one can ask for fixed $\beta > 0$, letting $n \to \infty$ with $m = \lfloor \beta n \rfloor$, is there a limiting probability of satisfiability and what is it. For $k = 2$ it has recently been shown (in [dlV92],[CR92], and by A. Goerdt [dlV]) that the threshold constant is 1; it remains to analyze what happens for m near n. For $k > 2$ it is not known whether threshold a constant exists; for some

$c > 0$ it has recently been shown (in [CR92])that for $m/n \leq c\,2^k/k$ the formula is satisfiable with high probability (which one can compare with the obvious threshold upper bound of $\log_{2^k/(2^k-1)} 2$); it is also known that the threshold ratio upper bound of $\log_{8/7} 2$ for 3-SAT can be improved upon slightly (by de la Vega, [dlV]).

Here we study the coefficients c_i resulting from proposition 2.2, and we write c_i^n or $c_i^{n,k}$ to indicate the dependence on n, k. Since a non-trivial disjunctions of k variables is 0 precisely on a subcube of codimension k of $\mathbf{B}^n$, we are studying the complexes $\mathcal{A}_i^{n,k}$ of subsets of $\mathbf{B}^n$ which contain at most i codimension k subcubes.

For $k = 1$ one can easily write down an explicit formula for the c_i's:

Theorem 3.1 *For $k = 1$ and any n we have*

$$c_i = \begin{cases} (-1)^{n+1}(-2)^i\binom{n}{i}, & \text{if } i \leq n, \\ 1, & \text{if } i = 2n, \\ 0. & \text{otherwise.} \end{cases}$$

Proof For $A \subset \mathbf{B}^n$ let $\iota_n(A)$ denote the number of codimension 1 cubes it contains. For $A, B \in \mathbf{B}^{n-1}$, let $A * B$ denote the subset of $\mathbf{B}^n$ whose restriction to $x_n = 0, 1$ is, respectively, A, B. We have

$$\iota_n(A * B) = \iota_{n-1}(A \cap B) + \left\{ \begin{array}{ll} 0, & \text{if neither } A \text{ nor } B \text{ are } \mathbf{B}^{n-1} \\ 1, & \text{if exactly one of } A, B \text{ are } \mathbf{B}^{n-1} \\ 2, & \text{if both } A \text{ and } B \text{ are } \mathbf{B}^{n-1} \end{array} \right\}. \tag{3.1}$$

The above allows us to determine the c_i's, analyzing each of the three cases for A, B seperately, viewing $C = A \cap B$ as fixed. For example, for the first case one observes

$$\sum_{\substack{\{(A,B)|C=A\cap B,\\ A\neq\mathbf{B}^n\neq B\}}} (-1)^{|A*B|} = \sum_{A,B}(-1)^{|A|+|B|} = \sum_A \left\{ \begin{array}{ll} 0 & \text{if } A \neq C \\ -1^{|A|+1} & \text{if } A = C \end{array} \right\} = (-1)^{|C|+1}. \tag{3.2}$$

The above sum contributes to c_i for every C with $\iota(C) = i$. Analyzing the other two cases yields $c_i^n = -c_i^{n-1} + 2c_{i-1}^{n-1}$ for $i \leq n - 1$, from which the theorem easily follows.

□

For $k = 2$ we do not know of a simple formula for (all) the c_i's. A computer-aided calculation gives the following values of $c_0, \ldots, c_N$, $N = 4\binom{n}{2}$:

$n = 2$: $c_0 = 1, c_1 = -4, 6, -4, c_4 = 1$
$n = 3$: $3, -12, 6, 24, -18, -24, 16, 12, 0, -8, 0, 0, 1$
$n = 4$: $15, -72, 84, 48, -84, 0, -152, 192, 0, 128, -192, 24, -56, 48, 0, 32, 0, 0, -16, 0, 0, 0, 0, 0, 1$
$n = 5$: $105, -600, 1020, 0, -1500, 1440, -1320, 1280, -10, -1040, 2760, -3360, 640, 960, -2080,$
$3000, -1120, -400, 960, -960, 96, 240, -320, 160, 0, 0, 80, 0, 0, 0, -32, 0, .., 0, c_{40} = 1$

Of course, the c_i with i large, say $i \geq$ roughly $3N/4$ are easy to determine, because they result only from $R \subset \mathbf{B}^n$ of small size (i.e. $|R| = 1, 2$). More interesting is the pattern of the c_0's, which suggests:

$$c_0^{n,2} = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 3). \tag{3.3}$$

We will prove this and derive formulas for c_1, c_2 as well in sections 6 and 7. It would be intersting to find general formulas for the c_i's and to have other proofs of formulas like the above; perhaps there is a way to prove the above by relating $\mathcal{A}_0^{n+1}$ to $\mathcal{A}_0^n$ in some geometric way (e.g. via a covering).

As an example, consider $\mathcal{A}_0^{3,2}$, the complex of subsets of $\mathbf{B}^3$ which contain no adjacent vertices. There are six maximal faces: two containing four vertices (corresponding to $\mathbf{B}^3$ points of XOR 0 or 1), and four contain two vertices (the four pairs of antipodal points). It not only follows that $-\tilde{\chi}(\mathcal{A}_0^{3,2}) = 3$, but that $\mathcal{A}_0^{3,2}$ has the homotopy type of a boquet of three circles, and so all but one of the reduced betti number vanishes. This purity of the reduced homology is true in many (but not all) cases computed by the author, and will be proven for $k = 1$, i.e. for all $\mathcal{A}_i^{n,1}$, in the next section.

Note that for large n almost all subsets of $\mathbf{B}^n$ contain no codimension 2 subcube, and so c_0 is a sum over $(-1)^{|R|}$ of almost all of the 2^{2^n} subsets of $\mathbf{B}^n$. Equation 3.3, and in general these calculations, suggest that there is a great deal of cancellation occurring in these coefficients.

We also give a calculation of 3-SAT coefficients, although at present we can say little about them (except for $c_i^{n,3}$ with $i \geq$ roughly $7\binom{n}{3}$):

$n = 3$: $c_0 = 1, c_1 = -8, 28, -56, 70, -56, 28, -8, c_8 = 1$

$n = 4$: $7, -32, 64, -288, 976, -928, -960, 928, 2396, -1536, -2144, -352, 2184, 1312,$
$-544, -1568, -460, 384, 736, 256, -184, -256, -96, 0, 88, 32, 0, 0, -16, 0, 0, 0, 1$

$n = 5$: $57, -720, 4800, -21840, 76340, -181664, 218840, -4400, -262020, 273600, -290016,$
$188880, 310620, -272960, -101440, -112224, -12040, 23920, 410960, 36000, -148952,$
$-113040, -238240, 35200, 78130, 53664, 131840, -13280, 71600, -90480, -8216, -90080,$
$-5920, -30720, 57880, 7296, 27920, 8320, 11840, -5920, -15880, 1920, -9480, -6080,$
$-6920, 6720, 1520, 960, 2760, 2880, 720, -1600, 800, -320, -800, -880, 0, 0, -320, 0, 256,$
$160, 0, 0, 80, 0, 0, 0, 0, 0, -32, 0, \ldots, 0, c_{80} = 1$

The explicit formula for $c_i^{n,1}$ can be used to calculate the satisfiability probability in the interesting range of m. For $k = 2$ the behavior of m near the threshold range of $1 \cdot n$ is unknown, and a study of the above coefficients may yield results. Of course, calculating the probabilities using the c_i's and equation 2.2 involves a lot of cancellation, and so it does not suffice to merely get good estimates on the $c_i^{n,k}$ (although this would be an interesting intermediate step for $k \geq 2$).

Proposition 3.2 *For any fixed $\beta > 0$, taking $n \to \infty$ and $m = \lfloor \beta n \rfloor$, the probability of unsatifiability tends to the limit*

$$\left(1 - e^{-\beta^2/4}\right)$$

Proof By the above this probability is

$$1 + \sum_{i=0}^{n} (-1)^{n+1} (-2)^i \binom{n}{i} \left(\frac{i}{2n}\right)^m = 1 + \frac{(-1)^{n+1}}{(2n)^m} \sum_{i=0}^{n} (-2)^i \binom{n}{i} i^m.$$

Coverting the i^s's to a sum of $i^{(j)}$ with $0 \leq j \leq s$ via Stirling numbers of the second kind, τ_k^n (recall $x^{(k)} = k!\binom{x}{k}$), the above follows easily using $\tau_{m-k}^m = m^{2k}/2^k k! + O(m^{k-1})$; see appendix A for details.

4 A purity theorem for 1-SAT coefficients and some homology calculations

In order to better understand the $c_i^{n,k}$ and possible relationships among them, we consider the complexes $\mathcal{A}_i^{n,k}$. While it is often the case that abstract simplicial complexes arising in combinatorics are homopotic to a boquet of spheres, and therefore have all but one of their reduced betti numbers vanishing, these complexes usually have some nice properties, such as the dimension of the maximal faces are constant (and equal to the dimension of the spheres of which it is a boquet, up to homotopy). Although this is not true for almost all of the $\mathcal{A}_i^{n,k}$, we still have purity for $k = 1$:

Theorem 4.1 $\tilde{H}_j(\mathcal{A}_i^{n,1}, \mathbf{Z}) = 0$ *for* $j \neq 2^n + i - n - 2$.

Question 4.2 *For which* k, n, i *is there purity in the reduced homology of* $\mathcal{A}_i^{n,k}$*? In those cases, does* $\mathcal{A}_i^{n,k}$ *have the homotopy type of a boquet of spheres?*

We note that purity does not hold in general, as can be verified by computer-aided calculations. For $k = 2, n = 3$, we have that purity holds for the reduced betti numbers in all cases except $i = 4$, where $\tilde{h}_3 = 7$ and $\tilde{h}_4 = 4$. For $k = 2, n = 4$, the author has verified purity in the case $i = 0, 1$, where all but respectively $\tilde{h}_7, \tilde{h}_8$ vanish, and that it fails in $i = 2$, where both $\tilde{h}_8, \tilde{h}_9$ don't vanish.

Proof (Of theorem 4.1.) The heart of the theorem is to develop a homology analogue to equation 3.2 and similar such equations (and to induct on n). So fix a set T (eventually we will take $T = \mathbf{B}^{n-1}$) with t elements. As before, we define $A * B$ to be the disjoint union of A and B (e.g. $\mathbf{B}^{n-1} * \mathbf{B}^{n-1}$ was earlier viewed as $\mathbf{B}^n$); $*$ extends to an operation on simplicial complexes, called the *join*. For a $C \subset T$ consider the collections of subsets of $T * T$ defined by

$$\mathcal{R}(C) = \mathcal{R}_T(C) \stackrel{\text{def}}{=} \{A * B | A, B \subset T, A \cap B = C\},$$

and define $\mathcal{R}'$ similarly but requiring both A and B to be proper subsets of T. For an abstract simiplicial complex, $\mathcal{C}$, of T define $\mathcal{R}(\mathcal{C})$ to be the union over all $\mathcal{R}(C)$'s with $C \in \mathcal{C}$, and define $\mathcal{R}'(\mathcal{C})$ similarly; both are simplicial complexes. The main tool we develop is:

Theorem 4.3 $\mathcal{R}, \mathcal{R}'$ *shift reduced homology by* $+t, +t-1$ *respectively, i.e.* $\tilde{H}_{j+t}(\mathcal{R}(\mathcal{C})) = \tilde{H}_j(\mathcal{C})$ *and* $\tilde{H}_{j+t-1}(\mathcal{R}(\mathcal{C})) = \tilde{H}_j(\mathcal{C})$, *for all* j *and* $\mathcal{C}$.

We first describe how to complete the proof of theorem 4.1 based on this theorem. The analogue of equation 3.1 is that for $i \leq n-1$ we have

$$\mathcal{A}_i^n = \mathcal{R}'(\mathcal{A}_i^{n-1}) \cup \mathcal{R}(\mathcal{A}_{i-1}^{n-1}),$$

where we have omitted the $k = 1$ superscripts in all of the above $\mathcal{A}$'s. Noticing that

$$\mathcal{R}'(\mathcal{A}_i^{n-1}) \cap \mathcal{R}(\mathcal{A}_{i-1}^{n-1}) = \mathcal{R}'(\mathcal{A}_{i-1}^{n-1}),$$

yields the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_j\Big(\mathcal{R}'(\mathcal{A}_{i-1}^{n-1})\Big) \to \tilde{H}_j\Big(\mathcal{R}'(\mathcal{A}_i^{n-1})\Big) \oplus \tilde{H}_j\Big(\mathcal{R}(\mathcal{A}_{i-1}^{n-1})\Big) \to \tilde{H}_j\Big(\mathcal{A}_i^n\Big) \to \tilde{H}_{j-1}\Big(\mathcal{R}'(\mathcal{A}_{i-1}^{n-1})\Big) \to \cdots.$$

The theorem follows by induction, applying theorem 4.3, and verifying by hand, say, the case $n = 1$.

Proof (Of theorem 4.3.) First we check the theorem for $\mathcal{C}$ being the empty complex or the complex generated by one subset. The general case follows from this by considering the covering induced on $\mathcal{R}(\mathcal{C})$ or $\mathcal{R}'(\mathcal{C})$ induced by the covering of $\mathcal{C}$ by its maximal faces, and using the generalized Mayer-Vietoris principle (which is the spectral sequence associated to the induced Čech double complex). For details, see appendix A.

5 More on Complexes

Let S be a fixed set of size $s > 0$, and let $\mathcal{P}(S)$ denote the set of all subsets of S (including $\emptyset$ and S). For $\mathcal{A} \subset \mathcal{P}(S)$, we set

$$\nu(\mathcal{A}) = \sum_{A \in \mathcal{A}} (-1)^{|A|}.$$

If $\mathcal{A}$ is closed under taking subsets, we say $\mathcal{A}$ is an *abstract simplicial complex* or an *ideal* or a *decreasing system* on S; similarly for closure under supersets and *increasing system*. When $\mathcal{A}$ is an abstract simplicial complex, ν is minus the reduced Euler characteristic of any realization of $\mathcal{A}$, i.e. $\nu = -\tilde{\chi}$ in the previous notation. If $s \in S$ has the property that $A \in \mathcal{A}$ iff $A - \{s\} \in \mathcal{A}$, we say s is *irrelevant* to $\mathcal{A}$, in which case obviously $\nu(\mathcal{A}) = 0$.

Given $\mathcal{U} \subset \mathcal{P}(S)$ we define

$$\mathcal{U}^{\leq i} \stackrel{\text{def}}{=} \{B \subset S | B \text{ contains } \leq i \text{ elements of } \mathcal{U} \},$$

and we say that $\mathcal{U}$ is an *overcover* of $\mathcal{A}$ when $\mathcal{A} = \mathcal{U}^{\leq 0}$. We will be interested in computing ν of the $\mathcal{U}^{\leq i}$'s for certain $\mathcal{U}$'s (such as those described in sections 3 and 4).

Given $\mathcal{U} \subset \mathcal{P}(S)$, we define $\mathrm{cov}_{\mathcal{U}}$ to be the collection of subsets $\mathcal{W} \subset \mathcal{U}$ such that $\cup\mathcal{W} = S$, where $\cup\mathcal{W}$ denotes the union of all elements of $\mathcal{W}$, i.e. $= \cup_{W\in\mathcal{W}} W$, a subset of S; $\mathrm{cov}_{\mathcal{U}}$ is obviously an increasing system on $\mathcal{U}$.

Proposition 5.1 *$\nu(\mathrm{cov}_{\mathcal{U}}) = (-1)^s \nu(\mathcal{A})$ for $\mathcal{A} = \mathcal{U}^{\leq 0}$.*

Proof Follows easily from inclusion-exclusion, namely

$$\nu(\mathcal{A}) = -\sum_{B \notin \mathcal{A}} (-1)^{|B|} = - \sum_{B \supset \text{ at least one } U \in \mathcal{U}} (-1)^{|B|} = - \sum_{\mathcal{W} \subset \mathcal{U}, \mathcal{W} \neq \emptyset} (-1)^{|\mathcal{W}|+1} \sum_{\{B | \cup\mathcal{W} \subset B\}} (-1)^{|B|} =$$

$$\sum_{\mathcal{W} \subset \mathcal{U}, \mathcal{W} \neq \emptyset} (-1)^{|\mathcal{W}|} \left\{ \begin{array}{ll} (-1)^s & \text{if } \cup\mathcal{W} = S \\ 0 & \text{otherwise} \end{array} \right\} = \nu(\mathrm{cov}_{\mathcal{U}})(-1)^s.$$

□

The above proposition enables us to calculate $\nu(\mathcal{U}^{\leq i})$ from data on $\mathrm{cov}_{\mathcal{U}}$ as follows. For $i \geq 1$ we define $S^{(i)}$ to be the set of all subsets of S of size i (in particular $S^{(i)} \subset \mathcal{P}(S)$ and $S^{(1)}$ is canonically isomorphic to S). Given $\mathcal{A} \subset \mathcal{P}(S)$, we define $\mathcal{A}^{(i)}$ to be the subsets $T \subset S^{(i)}$ whose union of elements is a set in $\mathcal{A}$.

Theorem 5.2

$$\nu(\mathcal{A}^{(i)}) = \sum_{J\in\mathcal{A}} (-1)^{|J|-i+1}\binom{|J|-1}{i-1}.$$

Proof It obviously suffices to prove the formula for $\mathcal{A}$ consisting of one subset of S, which reduces to:

Lemma 5.3 *Let* $\mathcal{U} = S^{(i)}$, $\mathcal{A} = S$. *Then*

$$\nu(\mathcal{A}^{(i)}) = f(s,i) \stackrel{\text{def}}{=} (-1)^{s-i+1}\binom{s-1}{i-1}.$$

Proof We argue by fixing i and inducting on s. If $s \le i$ the formula clearly holds. If $A \subset S^{(i)}$ does not cover S, then it covers exactly one proper subset of S, and the sum of $(-1)^{|A|}$ over all A covering a proper subset $T \subset S$ is clearly, by induction, $f(|T|, i)$. Hence

$$\nu(\mathcal{A}^{(i)}) = -\sum_{A\notin \mathrm{cov}_\iota} (-1)^{|A|} = -\sum_{t=0}^{s-1}\binom{s}{t} f(t,i)$$

(where in the above sum we can omit $1 \le t \le i-1$, but not $t = 0$ which contributes 1 for $\emptyset \subset I$); the above equals $f(s,i)$ by a standard summation formula for binomial coefficients (see, e.g., [Knu73], page 58).

□

To apply the above, consider the natural map $\iota{:}\,\mathcal{U}^{(i)} \to S$, namely $\mathcal{W} \mapsto \cup\mathcal{W}$, and $\mathcal{B}^i = (\mathrm{cov}_\mathcal{U})^{(i)}$, which is an increasing system on $\mathcal{U}^{(i)}$; clearly $\iota(\mathcal{B}^i)$ is an overcover of $\mathcal{U}^{\le i-1}$. (Notice that $\iota(\mathcal{B}^i)$ may be a multiset, i.e. ι restricted to $\mathcal{B}^i$ may map two elements to the same set; this is no problem since all the above generalizes trivially to multisets.) We get:

Corollary 5.4

$$\nu(\mathcal{U}^{\le i-1}) = (-1)^s \sum_{\mathcal{W}\in \mathrm{cov}_\mathcal{U}} f(|\mathcal{W}|, i).$$

We finish this section by making a few more remarks concerning proposition 5.1. The proposition can be viewed as follows: given $\mathcal{U} \subset \mathcal{P}(S)$, we define $\mathrm{emp}_\mathcal{U}$ to be the collection of subsets $\mathcal{W} \subset \mathcal{U}$ such that $\cap\mathcal{W} = \emptyset$, and we say that $\mathcal{U}$ *covers* $\mathcal{A} \subset \mathcal{P}(S)$ if $\mathcal{U} \subset \mathcal{A}$ and $\cup_{U\in\mathcal{U}}\mathcal{P}(U) = \mathcal{A}$. The proposition is equivalent to the fact that $\nu(\mathcal{A}) = -\nu(\mathrm{emp}_\mathcal{U})$ when $\mathcal{U}$ is a cover of $\mathcal{A}$. (As such it is a special case of the Čech cohomology computation.)

We remark that given $\mathcal{A}$, $\mathcal{U}$ is an overcover of $\mathcal{A}$ iff $\mathcal{U}$ is disjoint from $\mathcal{A}$ and contains all the minimal (with respect to inclusion) subsets of S not in $\mathcal{A}$; similarly $\mathcal{U}$ is a cover of $\mathcal{A}$ iff $\mathcal{U}$ is a subset of $\mathcal{A}$ and contains all the maximal elements of $\mathcal{A}$.

Finally we remark that since $\mathrm{cov}_\mathcal{U}$ and $\mathrm{emp}_\mathcal{U}$ are closed under taking supersets, their complements are abstract simplicial complexes. This leads to an infinite family of simplicial complexes with the same (up to sign) value of ν; in the case of evaluating 2-SAT coefficients we have only found two members of this family to be useful.

6 2-SAT coefficients: c_0 and c_1

With notation as in the previous section, we will take $S = \mathbf{B}^n$ for some value of n, and let $\mathcal{U}$ be the collection of codimension 2 subcubes of $\mathbf{B}^n$ (we write $\mathcal{U}^n$ to emphasise the dependence on n when needed). So $\mathcal{U}^{\leq i}$ is just the $\mathcal{A}_i^{n,2}$ of section 3.

Theorem 6.1 *For any $n \geq 2$,*

$$c_0^{n,2} = \nu(\mathcal{A}_0^{n,2}) = \nu(\mathcal{U}^{\leq 0}) = \nu(\mathrm{cov}_{\mathcal{U}^n}) = 1 \cdot 3 \cdot \cdots \cdot (2n-3).$$

Proof It suffices to show the last inequality, which we do by induction on n. For $n = 2$ this can be verified by hand. For $n > 2$, fix coordinates $x_1, \ldots, x_n$ on $S = \mathbf{B}^n$, and let $\mathcal{U} = \mathcal{P}_0 \amalg \mathcal{P}_1 \amalg \mathcal{O}$, where $\mathcal{P}_i$ are subcubes where $x_n = i$, and $\mathcal{O}$ are subcubes not involving x_n. We have

$$\nu(\mathrm{cov}_{\mathcal{U}}) = \sum_{P_0 \cup P_1 \cup O = S} (-1)^{|P_0|+|P_1|+|O|},$$

where the summation extends over all $P_i \subset \mathcal{P}_i$, $O \subset \mathcal{O}$. Consider for fixed P_i the contribution to the RHS (right-hand-side) of the above for all relevant $O \subset \mathcal{O}$. Each element of $\mathcal{P}_i$ can be viewed as a codimension one subcube of $\mathbf{B}^{n-1}$, and $\mathcal{O}$ elements as subcubes of codimension 2 of $\mathbf{B}^{n-1}$; if we fix P_0, P_1, letting C_i, $i = 0, 1$ being the set of points in $\mathbf{B}^{n-1}$ not covered by the elemnts of P_i, then C_i are each the empty set or a subcube of codimension $|P_i|$, and we have $P_0 \cup P_1 \cup O = S$ iff O, viewed as a subcube of $\mathbf{B}^{n-1}$, covers $C_0 \cup C_1$. In particular, if $C_0 \cup C_1$ is not the entire $\mathbf{B}^{n-1}$, then clearly it does not intersect at least one subcube of codimension 2, $U \in \mathcal{O}$, and this U is irrelevant to the O's satisfying $P_0 \cup P_1 \cup O = S$ for those fixed P_1, P_2; in this case there is no contribution to the RHS of the above. Otherwise $C_0 \cup C_1 = \mathbf{B}^{n-1}$, and the contribution to the RHS is clearly $(-1)^{|P_0|+|P_1|}\nu(\mathrm{cov}_{\mathcal{U}^{n-1}})$.

It remains to see how we can have $C_0 \cup C_1 = \mathbf{B}^{n-1}$, which can be broken down into several cases. The first way is to have $C_0 = \mathbf{B}^{n-1}$ and C_1 anything but $\mathbf{B}^{n-1}$, meaning $P_0 = \emptyset$, and $P_1 \neq \emptyset$; the total contribution of this is

$$\nu(\mathrm{cov}_{\mathcal{U}^{n-1}}) \sum_{P_1 \subset \mathcal{P}_1, P_1 \neq \emptyset} (-1)^{|P_1|} = \nu(\mathrm{cov}_{\mathcal{U}^{n-1}})(-1).$$

The second way is the reverse, which gives the same contribution. The third way is $P_0 = \emptyset = P_1$, contributing $\nu(\mathrm{cov}_{\mathcal{U}^{n-1}})$. The only other way one can have $C_0 \cup C_1 = \mathbf{B}^{n-1}$ is for C_0, C_1 to be complementary half-cubes; this can happen in $2(n-1)$ ways, each contributing $\nu(\mathrm{cov}_{\mathcal{U}^{n-1}})$. The total sum is therefore

$$\nu(\mathrm{cov}_{\mathcal{U}^{n-1}})(2n-3).$$

□

In what follows we derive a formula for $\nu(\mathcal{A}_1^{n,2})$, in a similar but much messier analysis.

Let $\mathcal{B}^{2,n}$ be $(\mathrm{cov}_{\mathcal{U}})^{(2)}$, so that $\nu(\mathcal{B}^{2,n}) = \nu(\mathcal{A}_1^{n,2})$, which we denote $\nu(n)$. Given a subcube U of $\mathbf{B}^n$ of codim 2, let $\mathcal{B}_U^{2,n}$ be the subset of $\mathcal{U}^{(2)}$ consisting of $\mathcal{W}$ with $U \cup \bigcup \mathcal{W} = S$; clearly ν of this complex is independent of which U is chosen, and we denote this value by $\tilde{\nu}(n)$. $\mathcal{U}^{(2)}$ is the union of $\mathcal{P}_i^{(2)}, \mathcal{O}^{(2)}$ and "cross terms" $\mathcal{P}_0 - \mathcal{P}_1, \mathcal{P}_0 - \mathcal{O}, \mathcal{P}_1 - \mathcal{O}$ where $\mathcal{A} - \mathcal{A}'$ denotes pairs with one element from each of $\mathcal{A}, \mathcal{A}'$, and have

$$\nu(\mathcal{B}^{2,n}) = \sum_{P_0 - P_0 \in \mathcal{P}_0 - \mathcal{P}_0, P_0 - P_1 \in \mathcal{P}_0 - \mathcal{P}_1, \ldots, O - O \in \mathcal{O} - \mathcal{O}} (-1)^{|P_0 - P_0| + |P_0 - P_1| + \cdots + |O - O|},$$

where $P_0 - P_1$ denotes a typical element of $\mathcal{P}_0 - \mathcal{P}_1$, etc. Consider the contribution to the RHS of the above with all but $O - O$ fixed. Letting C_0, C_1 be as before, i.e. C_0 being the points of $\mathbf{B}^{n-1}$ not hit by any $U \in \mathcal{P}_0$ coming from a $\mathcal{P}_0 -$ (anything) pair, and letting D be the points hit by $U \in \mathcal{O}$ coming from $\mathcal{P}_i - \mathcal{O}$ pairs, we see that the RHS contribution summing over all relevant $O - O$'s is either $\nu(n-1)$, $\tilde{\nu}(n-1)$, or 0, according to whether D union the complement of $C_0 \cup C_1$ is (1) empty, (2) a subcube of codimension 2, (3) anything else; the point is that anything else necessarily contains two distinct elements of $\mathcal{O}$, yielding an irrelevant element of $\mathcal{O}^{(2)}$. A similar remark holds for $\nu(\mathcal{B}_U^{2,n})$ assuming U is taken independent of x_n, the contribution to the RHS being either $\tilde{\nu}(n-1)$ or 0. A messy case anaylsis yields:

Theorem 6.2 *$\nu(2) = -3$, $\tilde{\nu}(2) = -1$, and for all $n \geq 3$,*

$$\nu(n) = (1-2n)\nu(n-1) + 4(2n-3)(n-1)(n-2)\tilde{\nu}(n-1), \qquad \tilde{\nu}(n) = (2n-5)\tilde{\nu}(n-1).$$

In other words, for all $n \geq 4$ we have

$$\nu(n) = (1-2n)\nu(n-1) - 8\binom{n-1}{2}(2n-3)[1 \cdot 3 \cdot \cdots \cdot (2n-7)]$$

$$= (-2n^2 + 4n - 3)[1 \cdot 3 \cdot \cdots \cdot (2n-5)],$$

and

$$c_1(n) = -2n(n-1)[1 \cdot 3 \cdot \cdots \cdot (2n-5)].$$

Proof The values of $\nu(2), \tilde{\nu}(2)$ can be checked by hand. The proof of the recurrences is a messy case analysis, whose details are given in appendix B.

□

7 2-SAT coefficients: c_2 and higher coefficients

In this section we reorganize the method for computing c_0, c_1 into one which enables us to compute c_2 and shows that c_i is given by a system of linear recurrences. Let $\mathcal{B}^{k,n}$ be $(\mathrm{cov}_{\mathcal{U}})^{(k)}$ with $\mathcal{U}$ as before. Fix k, and set $\nu(n) = \nu_0(n) = \nu(\mathcal{B}^{k,n})$.

Theorem 7.1 *For any k there exists an integer I and a collection of integers $\nu_1(n), \ldots, \nu_I(n)$ related by the recurrences:*

$$\nu_i(n+1) = \sum_{j \geq i} P_{ij}(n)\nu_j(n)$$

for $i = 0, \ldots, I$ and all $n \geq 2k$, where $P_{ij}(n)$ are fixed polynomials. Furthermore, $P_{ii}(n)$ is either $\pm a_i$ or $\pm(2n + a_i)$ with a_i an odd integer.

This theorem, and the calculations for $k = 1, 2, 3$ suggests:

Question 7.2 *For any k is it true that*

$$\nu(n) = Q(n)\,[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2k-1)]$$

for some polynomial Q of degree $2k-2$ (and similarly for the 2-SAT coefficient $c_{k-1}(n)$)?

If we knew more about the P_{ij}'s and $\nu_i(n_0)$ for some n_0 we might be able to settle the above, which does hold for $k = 1, 2, 3$.

The general method used to prove the above theorem also permits us to calculate $c_2(n)$ without too much pain. In the below $\nu(n) = \nu(\mathcal{B}^{3,n})$, $\tilde{\nu}(n) = \nu(\mathcal{B}_U^{3,n})$ (with notation as before), and ν_3, ν_5 will be described later.

Theorem 7.3 *We have that for* $n \geq 4$,

$$\begin{aligned}
\nu(n+1) &= -\nu(n) + 8\binom{n}{2}(n-3)\tilde{\nu}(n) + 4\binom{n}{2}(3n-7)\nu_3(n) + 96\binom{n}{4}(3n-8)\nu_5(n) \\
\tilde{\nu}(n+1) &= (5-2n)\tilde{\nu}(n) + 4(n-2)\nu_3(n) + 4(2n-5)(n-2)(n-3)\nu_5(n) \\
\nu_3(n+1) &= (2n-3)\nu_3(n) \\
\nu_5(n+1) &= (2n-5)\nu_5(n)
\end{aligned}$$

and

$$\nu(4) = 27, \quad \tilde{\nu}(4) = 4, \quad \nu_3(4) = 3, \quad \nu_5(4) = 1.$$

In particular for $n \geq 3$ *we have*

$$\nu(n) = (2n^4 - 14n^3 + 33n^2 - 33n + 15)1 \cdot 3 \cdot \cdots \cdot (2n-7),$$

and

$$c_2(n) = n(n-1)(2n^2 - 8n + 7)1 \cdot 3 \cdot \cdots \cdot (2n-7).$$

Proof We begin by describing the general method. Fix k, consider all unions of $\leq k-1$ codimension 2 cubes, $\mathcal{W} \subset \mathbf{B}^n$. We say that $\mathcal{W}$ is similar to $\mathcal{W}'$ if there is a symmetry of $\mathbf{B}^n$ (i.e. a permutation of $\{x_1, \ldots, x_n\}$ followed by a map taking each x_i to x_i or $1-x_i$) mapping $\mathcal{W}$ to $\mathcal{W}'$. Clearly there are only finitely many similarity classes of $\mathcal{W}$'s (for arbitrary n, where a $\mathcal{W}$ in $\mathbf{B}^n$ determines a class in $\mathbf{B}^N$ for all $N \geq n$ in the obvious way), each one specifiable by the first $\leq 2k-2$ coordinates x_i (i.e. each $\mathcal{W}$ similar to one which is the union of cubes given by fixing two coordinates among $x_1, \ldots, x_{2k-2}$). Fix one representative for each similarity class, $\mathcal{W}_0 = \emptyset, \mathcal{W}_1, \ldots, \mathcal{W}_I$, and let $\nu_i(n) = \nu(\mathcal{B}_{\mathcal{W}_i}^{k,n})$, where $\mathcal{B}_{\mathcal{W}}^{k,n}$ denotes those subsets of $\mathcal{U}^{(k)}$ whose union with $\mathcal{W}$ covers all of $\mathbf{B}^n$. $\nu_i(n)$ is defined for all n big enough so that $\mathbf{B}^n$ contains a subset similar to $\mathcal{W}_i$, which is certainly true for $n \geq 2k-2$.

We begin by arguing that there is a recurrence for $\nu_0(n+1)$ in terms of $\nu_i(n)$. $\nu_0(n)$ is a sum over all k element multisets on $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{O}\}$; as before we view the $\mathcal{O}^{(k)}$ summation last and consider fixed subsets of the other multisets on $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{O}\}$. So consider all such fixed subsets (i.e. subsets of $\mathcal{U}^{(k)}$ avoiding $\mathcal{O}^{(k)}$) whose image in $\mathcal{P}_0, \mathcal{P}_1, \mathcal{O}$ is P_0, P_1, O respectively. By theorem 5.2 we have

$$\nu_0(n+1) = \sum_{P_0, P_1, O} (-1)^{|P_0|+|P_1|+|O|+k-1} \binom{|P_0| + |P_1| + |O| - 1}{k-1} \nu_{O \cup (P_0 \cap P_1)}(n) \tag{7.1}$$

where the ν in the above is the ν_i corresponding to the $\mathcal{W}_i$ in the similarity class of $O \cup (P_0 \cap P_1)$ (this always being a union of codim 2 subcubes), and the summation extending over all P_0, P_1, O with $O \cup (P_0 \cap P_1)$ a union of $\leq k-1$ subcubes. (In the above formula O consists of $\leq k-1$ subcubes, so the fact that we omit $\mathcal{O}^{(k)}$ does not affect theorem 5.2.)

We break the above sum into a number of cases. The first is $P_0 = P_1 = \emptyset$. In this case we only get a contribution when $|O| = 0$, yielding a contribution of $\nu_0(n)$.

The next case is $P_0 = \emptyset$, $P_1 \neq \emptyset$. The total contribution is

$$\sum_{p_1=1}^{|\mathcal{P}_1|} \sum_{O} (-1)^{p_1+|O|+k-1} \binom{p_1+|O|-1}{k-1} \binom{|\mathcal{P}_1|}{p_1} \nu_O(n);$$

since $\sum_{p_1=0}^{M}(-1)^{p_1}\binom{M}{p_1}Q(p_1)$ for any polynomial Q of degree $\leq M-1$, and since $|\mathcal{P}_1| \geq k$ assuming $n \geq 2k+1$, the above sum is just

$$-\sum_{O}(-1)^{|O|+k-1}\binom{|O|-1}{k-1}\nu_O(n),$$

which is just $-\nu_0(n)$ (again only $|O| = 0$ contributes).

The case $P_1 = \emptyset$, $P_0 \neq \emptyset$ gives the same contribution.

The only remaining cases are when both P_0, P_1 are nonempty. It suffices to consider the cases when each $|P_i|$ is at most k, for if one is larger and the other is nonempty then $P_0 \cap P_1$ is the union of at least k codim 2 subcubes; also one can assume $|O| \leq k-1$. Using the symmetries of $\mathbf{B}^n$, it suffices to consider a finite set of cases (namely those where each P_i is given by fixing x_{n+1} and one coordinate among $x_1, \ldots, x_{2k-2}$, and each O is determined by fixing two coordinates among the first $2k-2$). Each such case contributes a constant term (coming from $(-1)^{|P_0|+|P_1|+|O|+k-1}\binom{|P_0|+|P_1|+|O|-1}{k-1}$) times a ν_i term times the number of ways the term can arise from symmetries, which is clearly a polynomial in n. Hence we have

$$\nu_0(n+1) = P_{00}(n)\nu_0(n) + \cdots + P_{0I}(n)\nu_I(n)$$

for all $n \geq 2k$ with fixed polynomials P_{0j}.

The same argument shows that all the ν_i's satisfy such a recurrence. Clearly we have $P_{ij} = 0$ if $\mathcal{W}_i$ is not, after applying some symmetry of $\mathbf{B}^n$, a subset of $\mathcal{W}_j$. Ordering the $\mathcal{W}_i$ so that their size is non-decreasing with i gives $P_{ij} = 0$ for $i > j$.

Next we consider P_{00}. When both P_0, P_1 are nonempty, the only way that $O \cup (P_0 \cap P_1)$ can be empty is for $O = \emptyset$ and for P_0, P_1 to consist of complementary half-cubes (in their images in $\mathbf{B}^n$, which can happen in $2n$ ways); since $|P_0| + |P_1| + |O| = 2$ in this case, it only contributes for $k = 1, 2$. Hence we have $P_{00} = -1$ for $k \geq 3$ (and $= -1-2n, -1+2n$ for $k = 2, 1$).

Next consider P_{jj} for $j > 0$. The contribution to P_{jj} from the cases where one or both of P_i are empty is -1; the contribution when P_0, P_1 are each of size one and consist of complementary half-cubes contributes only when some O with $2 + |O| \geq k$ is a subset of $\mathcal{W}_i$, which occurs only when $\mathcal{W}_i$ consists of $k-2$ or $k-1$ codim 2 subcubes; in these cases the total contribution is easily seen to be $-2n, 2n$ respectively. When both P_0, P_1 are nonempty, aside from the case where P_0, P_1 consist of complementary half-cubes, there are only a finite number of other possibilities (truly finite, not merely up to $\mathbf{B}^n$ symmetry), since we must have $O \cup (P_0 \cap P_1) \subset \mathcal{W}_j$. In each of these finite cases we have $P_0 \neq P_1$, for otherwise $P_0 \cap P_1$ would contain a half-cube which contains $4\binom{n}{2}$ codim 2 cubes; hence the (P_0, P_1, O) possibilities can be paired off into pairs in which the first two members are switched. Hence P_{jj} is -1 plus an even constant and possibly plus $\pm 2n$.

In the rest of this section we summarize the $k = 3$ calculation, leaving the details for an appendix. Set $\nu, \tilde{\nu}$ as before, and set $\nu_i(n) = \nu_{\mathcal{W}_i}$ for $i = 1, \ldots, 5$, where $\mathcal{W}_1 = \{x_1x_2, x_1x_3\}$,

$\mathcal{W}_2 = \{x_1x_2, \overline{x_1}x_2\}$, $\mathcal{W}_3 = \{x_1x_2, \overline{x_1x_2}\}$, $\mathcal{W}_4 = \{x_1x_2, \overline{x_1}x_3\}$, $\mathcal{W}_5 = \{x_1x_2, x_3x_4\}$ (here x_ix_j denotes the subcube with $x_i = x_j = 0$, $x_i\overline{x_j}$ the subcube $x_i = 0, x_j = 1$, etc.). Clearly $\nu_2(n) = \nu_4(n) = 0$ for $n \geq 3$. A not too long analysis of the cases $(|P_1|, |P_2|) = (1,1), (2,1), (3,1), (2,2)$ (no others are of interest) yields the recursion

$$\nu(n+1) = -\nu(n) + 8\binom{n}{2}(n-3)\tilde{\nu}(n) + 4\binom{n}{2}(3n-7)\nu_3(n) + 96\binom{n}{4}(3n-8)\nu_5(n) \quad (7.2)$$

(see appendix C for details).

One can use the above to easily get the recurrences for the other ν's. For example, $\tilde{\nu}(n+1)$ equals the RHS of the above with the following substitutions:

$$\nu \mapsto \tilde{\nu}, \quad \tilde{\nu} \mapsto \frac{1}{4\binom{n}{2}}\tilde{\nu},$$

$$\frac{1}{4\binom{n}{2}}\nu_3, \quad \frac{4(n-2)}{4\binom{n}{2}}\nu_1, \quad \frac{4\binom{n-2}{2}}{4\binom{n}{2}}\nu_5.$$

This is because $\tilde{\nu} = \nu_U$, and we can consider all $4\binom{n}{2}$ images, W, of U under $\mathbf{B}^n$ symmetries and take $\tilde{\nu}(n+1)$ as the average of the ν_W's. Then the exact same analysis which yielded equation 7.2 can be applied, except that $\nu_{\mathcal{W}}(n)$ terms must be replaced by the average of $\nu_{\mathcal{W}\cup W}(n)$ averaged over all U images, W. For example, when $\mathcal{W} = U$, then finding the average of $\nu_{\mathcal{W}\cup W}(n)$ is a matter of determining for the $4\binom{n}{2}$ different W's what is the shape of $\mathcal{W} \cup W$. There is one W for which one gets U itself, one yielding $\mathcal{W}_3$, $4(n-2)$ yielding $\mathcal{W}_1$, etc.

This same averaging trick shows that $\nu_3(n+1)$ is the RHS of equation 7.2 with the substitutions:

$$\nu \mapsto \nu_3, \quad \nu_3 \mapsto \frac{1}{2\binom{n}{2}}\nu_3, \quad \tilde{\nu} \mapsto \frac{2}{4\binom{n}{2}}\nu_3,$$

all others mapped to zero, and similarly for ν_5 and ν_1. A calculation (which the author did by computer) using theorem 5.2, yields:

$$\nu(4) = 27, \quad \tilde{\nu}(4) = 4, \quad \nu_3(4) = 3, \quad \nu_5(4) = 1, \quad \nu_1(4) = 0,$$

from which the rest easily follows.

8 Betti number calculations

In performing numerical experiments, especially involving simplicial complexes on the set $\mathbf{B}^4$, we needed to develop fast methods of calculating betti numbers and/or checking if a certain betti number vanishes (if all but one of the betti numbers vanish, then the remaining one can be computed from the Euler characteristic, which is relatively very quickly computable).

The problem of computing homology with integer coefficients has received a fair amount of attention recently (see [DC91] and the references there). Computing the betti numbers, i.e. the rank of the homology groups, i.e. the dimension of the homology groups with real coefficients, would seem to be a much easier task since torsion is ignored. This task seems to have received little attention so far, but we suspect that in many applications (such as ours) it is sufficient to know the betti numbers, or at least interesting to know them when there are insufficient resources to compute homology over the integers.

Our observation is that if one is only interested in the betti numbers, working with homology with real coefficients, one can introduce Laplacians of the chain complex, which are integral, sparse, positive definite matrices vanishing on a space of dimension equal to the betti number. In particular, if the betti number is small or one only wants to check whether or not it vanishes, one can perform a simple (randomized) iterative method, which requires only linear space in $N =$ the maximum number of faces of a given dimension, as opposed to the quadratic space needed to compute the rank by the obvious method; in practice the amont of time needed is much smaller as well (we will explain why, although we do not attempt to formalize this here).

Recall that the betti numbers, h_i, are the dimensions of the homology groups, $H_i = \ker(\partial_i)/\mathrm{Im}(\partial_{i+1})$ of the chain complex,

$$\cdots \longrightarrow \mathcal{C}_{i+1} \xrightarrow{\partial_{i+1}} \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_0 = 0, \tag{8.1}$$

where $\mathcal{C}_i$ is the space of formal $\mathbf{R}$-linear sums of oriented i-dimensional faces, i.e. subsets of the abstract simplicial complex of size $i+1$, and ∂_i is the boundary map.

If the number of i-dimensional faces is d_i, then computing $h_i = \dim(H_i)$ naively involves computing the rank of a $d_{i+1} \times d_i$ matrix and of a $d_i \times d_{i-1}$ matrix. Also, typically one of $d_{i\pm 1}$ is considerably smaller than d_i and one is considerably larger (for practical matters, "considerably" is taken to mean a factor of 2 or more). Working with the Laplacian seems to greatly simplify the calculation.

Recall that for an arbitrary chain complex of vector spaces over $\mathbf{R}$ (or $\mathbf{Q}$ or $\mathbf{C}$ for that matter) as in equation 8.1, as soon as we endow each $\mathcal{C}_i$ with the structure of an inner product, we get maps $\partial_i^*: \mathcal{C}_{i-1} \to \mathcal{C}_i$ (i.e. the transpose of ∂_i), and thus a Laplacian (similar to [Wel80] or [GH78]), $\Delta_i: \mathcal{C}_i \to \mathcal{C}_i$, for each i, defined by

$$\Delta_i = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i.$$

For chain complexes where each $\mathcal{C}_i$ a finite dimensional $\mathbf{R}$-vector space, the analogue of harmonic theory (see [Wel80, GH78]) involves only elementary linear algebra, and says:

Proposition 8.1 (Harmonic theory) *For each i we have $\mathcal{C}_i$ decomposes into orthogonal subspaces*

$$\mathcal{C}_i = \ker(\Delta_i) \oplus \mathrm{Im}(\partial_{i+1}\partial_{i+1}^*) \oplus \mathrm{Im}(\partial_i^*\partial_i).$$

The Laplacian is positive definite on the latter two summands and is invariant on each. Elements of $\ker(\Delta_i)$ *are called the* harmonic forms, *and are in 1-1 correspondence with H_i (via $f \mapsto f$'s equivalence class in H_i). $\partial_{i+1}\partial_{i+1}^*$ is invariant on the middle summand and vanishes on the other two, and similarly for $\partial_i^*\partial_i$.*

In our case, $\mathcal{C}_i$ is simply $\mathbf{R}^{d_i}$, which we view as an inner product space with the standard Euclidean inner product. Then * is just the usual matrix transpose, and the ∂_i's, ∂_i^*'s, and Δ_i's are all integral matrices. The sum of any row of $\partial_i^*\partial_i$ is clearly bounded by $i(n-i+1)$, where n is the total number of vertices of the abstract simplicial complex, and similarly $\partial_{i+1}\partial_{i+1}^*$ row sums are bounded by $(i+1)(n-i)$. Since Δ_i is a symmetric matrix we have

$$\|\Delta_i\| \le i(n-i+1) + (i+1)(n-i).$$

Let B be a bound for $||\Delta_i||$, such as the right-hand-side of the above. Then $B - \Delta_i$ is a symmetric positive definite operator, its eigenvalues are all $\leq B$, and the eigenspace corresponding to the eigenvalue B is just the (possibly trivial) space of harmonic forms. We can calculate its largest eigenvalue(s) by the "power method" aka "simple vector iteration" method (see [SB80], page 354), namely by picking a random $v \in \mathbf{R}^{d_i}$ and repeatedly applying $B-\Delta_i$ to v and normalizing v. The result with almost surely converge to an eigenvector v_1 of largest eigenvalue. We can repeatedly generate more eigenpairs by choosing a random v, and then repeatly applying $B - \Delta_i$, projecting onto the subspace orthogonal to the previously discovered eigenvalues, and normalizing. This suggests the following randomized algorithm:

Algorithm 8.2 *To test whether or not $h_i = 0$, pick a random v and perform the simple vector iteration for $B - \Delta_i$. To calculate h_i when $h_i > 0$, continue the iteration method until an eigenvalue $< B$ occurs.*

Crutial to the above algorithm is that λ_1, the smallest non-zero eigenvalue of Δ_i, should not be very close to 0. We require the machine to have enough precision to distinguish B from $B - \lambda_1$. Also, under the simple vector iteration the component of the random vector which is orthogonal kernel of Δ_i will decay like $(1 - \lambda_1/B)^m$ where m is the number of iterations. So the larger λ_1/B, the better the convergence to an eigenvector; notice that one can experiment with small values of B (i.e. not known to be bounds for Δ_i) in the above method, and see if the limiting eigenvector has a positive eigenvalue with respect to $B - \Delta_i$; similarly, one can also replace B by anything larger than a bound for $||\Delta_i||/2$. The above suggests:

Question 8.3 *How small can λ_1 be? How small is λ_1 typically?*

The fact that λ_1 satisfies an algebraic equation whose coefficients can be bounded gives a very weak worst-case lower bound. For a worst-case upper bound, if $\mathcal{A}$ is a chain complex which is a cycle of length N (i.e. N maximal faces of size 2 connected to form one cycle), then λ_1 for $\mathcal{C}_1$ is clearly $2 - 2\cos(2\pi/n) \approx 2\pi^2/N^2$. On the other hand, calculations performed by the author (e.g. those described in §5) never produced a λ_1 as small as 1, and we suspect that typical λ_1's are much larger than worst-case λ_1's.

Of course, the above question restricted to complexes of dimension 2 is just a question about graphs, where much is known. We point out that λ_1 for "random" graphs is typically quite large (i.e. see [Fri91] and the references given there), while for many interesting graphs λ_1 is small. Thus it may not be interesting to look at only a naive notion of a random complexes.

It is important to note that in the above algorithm we do not need to have an apriori lower bound on λ_1 to detect the vanishing of a betti number. For if the iteration scheme produces a vector of Rayleigh quotient $B(1 - \eta)$ after m iterations, with $2m\eta$ sufficiently greater than $1/2 \log N - \log\log N$, then with "high probability" B cannot be an eigenvector. More precisely, we have:

Proposition 8.4 *Let A be a symmetric matrix with all eigenvalues non-negative, and largest eigenvalue B. If v is a vector such that $A^m v$ has Rayleigh quotient $\leq B(1-\eta)$, then $\log K - \log\log K + C \geq 2m\eta$ for a small constant C, where $K = K(v) =$ is the ratio of $||v||^2/||v'||^2$ where v' is the projection of v onto the eigenspace of B.*

For the proof and further discussion see the appendix. In the above, for a random vector v, $K(v)$ is typically roughly of size $\sqrt{N}$. In examples tested by the author, such as the complexes $\mathcal{A}_i^{4,2}$, $i = 0, 1, 2$, N was as large as 12,000 or so, and the smallest η encountered was roughly 1/20. In such cases slightly more than a hundred iterations suffices to check the non-vanishing of a betti number.

One might be concerned that the above is only a randomized algorithm. Assuming that we can distinguish B from $B - \lambda_1$, the above will always give a lower bound (almost surely an upper bound). If one has sufficient resources, one can obtain a randomized upper bound, or the exact answer, via:

Algorithm 8.5 *To calculate h_i, calculate the rank of Δ_i over $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. This rank is always $\leq d_i - h_i$, with equality holding for almost all p. If resouces permit, calculate the rank over $\mathbf{Z}$ (or over $\mathbf{F}_p$ for sufficiently many p) to get the exact answer.*

Note that this is an improvement over computing the ranks of ∂_{i+1} and of ∂_i whenever one of $d_{i\pm1}$ is substantially bigger than d_i. In practice the rank computing algorithms probably require quadratic space.

9 Concluding Remarks

We comment on further research in progress. Another situation in which the m-fold AND arises is in non-deterministic communication complexity, where the following question arises. Given a subcollection of entries, C, of an $r \times s$ matrix, how many submatrices which are subsets of C does it take to cover C? Here a submatrix is any subset of the matrix determined by specifying a subset of columns and a subset of rows. Letting $\mathcal{F}_1$ be the space of submatrices with uniform distribution (each of which can be viewed as a boolean function of rs variables), the question is for which m does the characteristic function of C first appear (i.e. occur with non-zero probability) in the m-fold AND of $\mathcal{F}_1$. Here one can write down a formula for the c_i's involving certain summations. The author has yet to investigate this matter further.

Another interesting question is can one evaluate the formulas for the event probabilities (perhaps times an appropriate integer) modulo p for some primes p? If the probability is nonzero when reduced modulo p, then one gets an upper bound for complexity. Perhaps this could be used to improve certain constructions based on counting arguments (which, as mentioned in the introduction, use only the first moments of the probability space). One might also be able to obtain lower bounds, but of course one would have to show the probability vanishes for a lot of p's, or else modulo p^m for one p but for a sufficiently large value of m.

One way of reducing modulo p, when the coefficients of the formula come from a geometric object, is to apply a p-group of symmetries to the object and to consider the orbits of size 1 (all others yield 0 modulo p). While a non-zero formula might vanish modulo p, a homology approach give more information. That is, assume the formula can be interpreted as a reduced Euler characteristic of a simplicial complex, and assume that one can prove that when the formula is zero the entire reduced homology necessarily vanishes. Then one can reduce the simplicial complex by the actions of cyclic groups, and any non-zero reduced homology in the reduced complex implies non-zero reduced homology in the original complex. This homological approach is more general than reducing the formula modulo a fixed prime, for one can apply a sequence of cyclic groups of arbitrary order. This approach has had some success in combinatorics, as in [KSS84]. This leads us to asking whether the above formulas can be interpreted as the Euler characteristic of a complex with the aforementioned property.

In view of the random 3-SAT and the suboptimal bound given by the first moment method, it seems natural to conjecture that other upper bounds given by first moment methods may not be optimal. Consider Valiant's monotone majority construction (see [Val84]). There he considers a $2d$-th level iterated NAND construction with $\mathcal{F}_1 = \{x_1, \cdots, x_n, 1\}$, with n odd, the x_i equally likely and 1 occuring with probability $.236067977\cdots$. He proves that for $d = (5.3\cdots)\log_4 n$ a random element of $\mathcal{F}_{2d}$ computes the majority function with high probability; he gives a first moment argument. It seems natural to conjecture that the true constant may be lower.

We mention that one can apply proposition 5.1 repeatedly, but that in this work only one application is used. For example, we use the fact that ν of $\mathcal{A}_0^{n,2}$ is the same as that of the family of all collections of codim 2 subcubes which cover $\mathbf{B}^n$. Applying the proposition once more yields the complex of all families of covering collections of codim 2 subcubes, having the property that the family contains each codim 2 subcube in at least one of its collections. The fact that ν of this complex is the same as the others is interesting, although the author has not found it to be of use in evaluating 2-SAT (or 3-SAT) coefficients.

We remark again that, as de la Vega points out (in [dlV92]), it would be interesting to understand random 2-SAT for m near n. The exact formula for $k=1$ is managable in the interesting range $m \approx \sqrt{n}$ because one can get a simple expression for $\sum_i c_i i^{(s)}$ for any s, n. Is the same true for 2-SAT?

Ultimately one would like to study this method for more exciting spaces, such as the iterated NAND space described in §2. But even the m-fold AND space has enough interest and mysteries at present to merit further study.

A Details of Proofs

Proof (Of proposition 3.2.) We simplify the sum by using Stirling numbers of the second kind (see [Knu73]), τ_k^n defined by $x^n = \sum_k \tau_k^n x^{(k)}$ with $x^{(k)} = k!\binom{x}{k}$. We have

$$\sum_{i=0}^{n}(-2)^i\binom{n}{i}i^m = \sum_{i,0\le k\le m}(-2)^i\binom{n}{i}i^{(k)}\tau_k^m = \sum_k \tau_k^m \sum_i \binom{n-k}{i-k}n^{(k)}(-2)^i = \sum_{k=0}^{m} n^{(k)}\tau_k^m(-2)^k.$$

The desired probability becomes

$$1 - \frac{1}{(2n)^m}\left(2^m n^{(m)}\tau_m^m - 2^{m-1}n^{(m-1)}\tau_{m-1}^m + \cdots\right). \tag{1.1}$$

For any fixed integer q we have

$$\frac{1}{(2n)^m}2^{m-q}n^{(m-q)}\tau_{m-q}^m = \frac{\tau_{m-q}^m}{(2n)^q}\Big(1+O(1/m)\Big) = \frac{\beta^{2q}m^{2q}}{2^q q!(2n)^q}\Big(1+O(1/m)\Big) = \frac{\beta^{2q}}{4^q q!}\Big(1+O(1/m)\Big),$$

by virtue of the fact that $\tau_{m-k}^m = m^{2k}/2^k k! + O(m^{k-1})$, which can be seen from the relation $\tau_{m-2}^m = (m-2)\tau_{m-2}^{m-1} + \tau_{m-3}^{m-1}$ (see [Knu73]). This last relation also implies that $\tau_{m-k}^m \le (m-2)\cdots(m+2k-3)/2^k k!$, from which it follows that for any $\epsilon > 0$ there is a Q such that truncating the series in equation 1.1 after Q terms introduces an error of no more than ϵ. It follows that the desired probability is within $O(1/m)$ of

$$1-\sum_{q=0}^{\infty}\frac{\beta^{2q}}{4^q q!}=\left(1-e^{-\beta^2/4}\right).$$

□

Proof (Of theorem 4.3.) In what follows, we write $A \sim B$ to mean that A and B have the same reduced homology, and we write S^d, B^d for the d-dimensional sphere and ball, respectively, and $\cdot = B^0$ for the point. Notice that $*$ is well defined modulo $\sim$, as follows from the (split) exact sequence

$$0 \to \tilde{H}_{p+1}(X * Y) \to \tilde{H}_p(X \times Y) \to \tilde{H}_p(X) \oplus \tilde{H}_p(Y) \to 0$$

(see [Mun84], page 373), and the Künneth formula (and its functoriality). Also note that $A * \cdot \sim \cdot$ for any A (which follows from the above or from the fact that $A * \cdot$ is a cone over A and hence acyclic). Also, it is well-known that $S^c * S^d$ is isomorphic to S^{c+d+1} (see [RS82] for example). In formulas involving abstract simplicial complexes and spheres or balls, the reader can identify the abstract simplicial complex with any realization of it; when dealing modulo $\sim$, the reader can alternatively can think of S^d or B^d as any fixed abstract simplicial complex with the same homology as the d-dimensional sphere, e.g. S^d as the set of all subsets of size $\le d+1$ of a set of $d+2$ elements.

We begin by proving theorem 4.3 in two simple cases. For any $C \subset T$, let $C^-, \mathcal{S}(C)$ denote the chain complexes of, respectively all subsets of C, and of all proper subsets of C.

Lemma A.1 *$\mathcal{R}(\emptyset), \mathcal{R}'(\emptyset)$ are respectively homeomorphic to $S^{t-1}, S^{t-2} \times [0,1]$. (I.e. any realization of $\mathcal{R}(\emptyset)$ is topologically homeomorphic to S^{t-1}, or there exists a triangulation of S^{t-1} whose vertices correspond 1-1 with $\mathcal{R}(\emptyset)$ and whose faces correspond; similarly for $\mathcal{R}'(\emptyset)$.)*

Proof $\mathcal{R}'(\emptyset)$ contains $\{A * \emptyset \mid |A| \le t-1\}$ and $\{\emptyset * A \mid |A| \le t-1\}\}$; each of these is isomorphic to a sphere S^{t-2}. The maximal faces of $\mathcal{R}'(\emptyset)$ are of the form $A*(T-A)$ ranging over $A \subset T$ with $1 \le |A| \le t-1$, which is easily seen to be a triangulation of $S^{t-2} \times [0,1]$, where $S^{t-2} \times \{0\}$ is the $\{A * \emptyset\}$ sphere and $S^{t-2} \times \{1\}$ is the antipodal map applied to the $\{\emptyset * A\}$ sphere. (In other words, we view S^{t-2} as the unit sphere about the origin in $\mathbf{R}^{t-1}$, fix points $p_1, \ldots p_t \in S^{t-2}$ which are the vertices of an equilateral simplex; consider the sphere of radius 2 about the origin and the equilateral simplex with vertices $-2p_1, \ldots, -2p_t$; etc.) $\mathcal{R}(\emptyset)$ is the union of $\mathcal{R}'(\emptyset)$ and the "polar caps," $T * \emptyset$ and $\emptyset * T$, which clearly yield the S^{t-1} sphere.

Lemma A.2 *For any proper subset C of T, $\mathcal{R}(C^-) \sim \mathcal{R}'(C^-) \sim \cdot$.*

Proof Viewing T as $C*(T-C)$, we see that $\mathcal{R}(C^-) = C^- * C^- * \mathcal{R}_{T-C}(\emptyset)$, recalling that $\mathcal{R}_{T-C}$ is $\mathcal{R}$ viewing things as subset of $T-C$. But C^- is isomorphic to the ball B^{c-1} with $c = |C|$, and is therefore $\sim \cdot$, assuming $C \ne \emptyset$, so $\mathcal{R}(C^-) \sim \cdot$. As for $\mathcal{R}'(C^-)$, the situation is more complicated. Viewing $\mathcal{R}'(C^-)$ as a complex over $C*C*(T-C)*(T-C)$ in the obvious way, we have that $\mathcal{R}'(C^-) = F_1 \cup F_2 \cup F_3$, where $F_1 = C^- * \mathcal{S}(C) * (T-C)^- * \emptyset$,

$F_2 = \mathcal{S}(C) * C^- * \emptyset * (T-C)^-$, and $F_3 = \mathcal{R}_C(\emptyset) * (T-C)^- * (T-C)^-$. We will calculate the homology of $\mathcal{R}'(C^-)$ by the above covering.

Clearly $F_i \sim \cdot$ for $i = 1,2,3$; we have $F_1 \cap F_2 = \mathcal{S}(C) * \mathcal{S}(C) * \emptyset * \emptyset \sim S^{c-2} * S^{c-2} \sim S^{2c-3}$. The Mayer-Vietoris sequence then gives $F_1 \cup F_2 \sim S^{2c-2}$. To analyze the homology of $(F_1 \cup F_2) \cap F_3$, we note that $F_i \cap F_3 \sim \cdot$ for $i = 1,2$, and $F_1 \cap F_2 \cap F_3 = F_1 \cap F_2 \sim S^{2c-3}$; the MV sequence for $(F_1 \cup F_2) \cap F_3 = (F_1 \cap F_3) \cup (F_2 \cap F_3)$ implies that it is $\sim S^{2c-2}$. Finally, the MV sequence for $F_1 \cup F_2 \cup F_3$ written as the union of $F_1 \cup F_2$ and F_3 then yields $F_1 \cup F_2 \cup F_3 \sim \cdot$, which is the desired result

□

Now consider a general simplicial complex $\mathcal{C}$ and its maximal faces, $M_1, \ldots, M_r$. We can compute the homology of $\mathcal{R}(\mathcal{C})$ from the Čech double complex (analogous to the Čech-de Rham double complex, as in [BT82], §8) resulting from the induced covering $\mathcal{R}(M_1^-), \ldots, \mathcal{R}(M_r^-)$ (which is not a "good" cover, i.e. some intersections of the $\mathcal{R}(M_i^-)$'s are not acyclic!). One can carry this out directly to prove the theorem (i.e. by induction on r, using the Mayer-Vietoris sequence), but it is perhaps easier to deduce the theorem by comparing the functors $\mathcal{C} \mapsto \mathcal{R}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathcal{C} * S^{t-1}$. The latter functor is well-known to shift reduced homology by t (see [Mun84], page 371), but one can also compute the homology of $\mathcal{C} * S^{t-1}$ by the induced covering, $M_1^- * S^{t-1}, \ldots, M_r^- * S^{t-1}$. For each functor the resulting Čech double complex (as in [BT82], §8, except with all arrows reversed), is the abutment of a spectral sequence (E_r, d_r) with $E_1^{p,q} = H_\partial^{p,q}$, ∂ being the boundary map on chains (as in [BT82], page 165, ∂ corresponding to d, and arrows reversed). By the above two lemmas the two E_1's are the same (i.e. each $E_1^{p,q}$ are isomorphic as $\mathbf{Z}$-modules), and from functoriality of the two functors it is clear that the d_1 coincide. Hence they abutt to the same limit, and so the two functors behave identically on homology.

The same proof holds for $\mathcal{R}'$, comparing it to $*S^{t-2}$.

□

Proof (Of proposition 8.4.) For $v \in \mathbf{R}^N$ with $K = K(v)$ as in proposition 8.4, it is easy to see that the Rayleigh quotient of $w = A^m v$ is

$$\frac{(Aw, w)}{(w, w)} \geq \min_{\lambda \geq 0} \frac{B + \lambda(\lambda/B)^{2m}(K-1)}{1 + (\lambda/B)^{2m}(K-1)}.$$

Denote the right-hand-side of the above by $f(\lambda)$. By differentiation we find that $f(\lambda)$ is minimized for the unique λ such that $f(\lambda) = (2m+1)\lambda/(2m)$, i.e. such that

$$\frac{2m+1}{2m}\lambda + \frac{1}{2m}\lambda^{2m+1}(K-1) = 1.$$

Setting $\lambda = 1 - x/(2m+1)$ and approximating e^{-x} for λ^{2m+1} yields

$$e^{-x} = \frac{x-1}{K-1},$$

from which we easily deduce the proposition (details omitted).

□

It should be noted that for a random choice of $v \in \mathbf{R}^N$ we have that $K(v)$ is typically proportional to $\sqrt{N}$, and that there is a constant C such that the probability that K exceeds $C\alpha\sqrt{N}$, for any $\alpha \geq 1$, is $\leq 1/\alpha$ (assuming any reasonable model of a "random" vector in $\mathbf{R}^N$, such as choosing each component with a normal distribution with mean 0, uniform distribution in $[-1, 1]$, etc.).

B Details of c_1 or $k = 2$ calculation

We begin with $\nu(n)$, i.e. evaluating $\nu(\mathcal{B}^{2,n})$.

1. $D = \emptyset$, i.e. $O - P_i = \emptyset$ for $i = 0, 1$.

   (a) $P_0 - P_0 \neq \emptyset$. Viewing the summation as first over $\mathcal{P}_1 - \mathcal{P}_1$ and then $\mathcal{P}_0 - \mathcal{P}_1$, it suffices to take $P_1 - P_1 = \emptyset$ (or else we get an irrelevant element of $\mathcal{P}_0 - \mathcal{P}_1$). We consider the subcases:

   i. $P_0 - P_1 = \emptyset$; here the total contribution is all variables fixed except for $P_0 - P_0 \neq 0$ (and the sum on $O - O$, of course). The $O - O$ sum for each $P_0 - P_0 \neq 0$ yields a $(-1)^{|P_0 - P_0|}\nu(n-1)$, and summing over all $P_0 - P_0 \neq 0$ gives a total of $-\nu(n-1)$.

   ii. $|P_0 - P_1| = 1$; we consider the P_0 subcube, U_0, and the P_1 subcube, U_1, of the $P_0 - P_1$ pair, viewing them as half-cubes of $\mathbf{B}^{n-1}$— as half-cubes they can be equal, complementary (i.e. disjoint), or skew (otherwise):

   A. Equal. This means that $C_0 \cup C_1$ is at most half of $\mathbf{B}^{n-1}$, so in all sums over $\mathcal{O} - \mathcal{O}$ there is an irrelevant element.

   B. Complementary. In this case the only way to avoid an $\mathcal{O} - \mathcal{O}$ irrelevant element is for $P_0 - P_0$ to consist of one pair, $\{U_0, U_0'\}$, with U_0' skew to U_0. The $P_0 - P_1$ piece can be any of $2(n-1)$, and U_0' can be any of $2(n-2)$ given U_0. Each contributes $\tilde{\nu}(n-1)$, since $C_0 \cup C_1$ is the complement of a codim 2 subcube of $\mathbf{B}^{n-1}$, for a total contribution of $4(n-1)(n-2)\tilde{\nu}(n-1)$.

   C. Skew. Again $P_0 - P_0$ must consist of the single element $\{U_0, U_0'\}$ with U_0' skew to U_0, for a total contribution of $4(n-1)(n-2)\tilde{\nu}(n-1)$ again.

   iii. $|P_0 - P_1| \geq 2$; to avoid $O - O$ irrelevance we must have $P_0 - P_1$ consisting of two elements, $\{U_0, U_1\}$ and $\{U_0', U_1\}$, with $P_0 - P_0$ consisting of the single element $\{U_0, U_0'\}$, such that U_0, U_1 are complementary and U_0' is skew to U_0. Similarly we get $-4(n-1)(n-2)\tilde{\nu}(n-1)$, the minus sign coming from the fact that this time $|P_0 - P_0| + |P_0 - P_1|$ is odd.

   (b) $P_0 - P_0 = \emptyset$, $P_1 - P_1 \neq \emptyset$— same as case (a).

   (c) $P_0 - P_0 = \emptyset = P_1 - P_1$. We divide into three cases:

   i. $P_0 - P_1 = \emptyset$. This obviously contributes a $\nu(n-1)$.

   ii. $|P_0 - P_1| = 1$; again we consider the single element, $\{U_0, U_1\}$ of $P_0 - P_1$, and consider the three possibilities for U_0, U_1:

   A. Equal. We get no contribution because of the irrelevant $\mathcal{O} - \mathcal{O}$ piece.

   B. Complementary. This can happen in $2(n-1)$, each contributing $-\nu(n-1)$, for a total of $-2(n-1)\nu(n-1)$.

C. Skew. This can happen in $2(n-1)2(n-2)$ ways, each contributing $-\tilde{\nu}(n-1)$, for a total of $-4(n-1)(n-2)\tilde{\nu}(n-1)$.

iii. $|P_0 - P_1| \geq 2$; to avoid $\mathcal{O}-\mathcal{O}$ irrelevance we must have $P_0 - P_1$ consisting of two elements, equal either $\{U_0, U_1\}, \{U_0', U_1\}$ or with 1 and 2 reversed; futhermore, in the former case, we must have U_0, U_0' skew $(2(n-1)(n-2)$ possibilities), and U_1 having two choices given U_0, U_0'. The total contribution is $8(n-1)(n-2)\tilde{\nu}(n-1)$.

2. D consists of one element of $\mathcal{O}$, which can happen in $2(n-1)(n-2)$ different ways $(=|\mathcal{O}|)$. To fix ideas say that D is the intersection of half-cubes U, U'. We consider a number of cases:

(a) $P_0 - P_0 \neq \emptyset$: summing lastly over $\mathcal{P}_0 - \mathcal{P}_1$ and $\mathcal{P}_0 - \mathcal{O}$ we must have $P_1 - P_1 = \emptyset = P_1 - O$ to avoid irrelevance. We consider the image, $\mathcal{I}_1$, of $P_0 - P_1$ in $\mathcal{P}_1$, i.e. the $\mathcal{P}_1$ elements occuring in $P_0 - P_1$:

i. $\mathcal{I}_1$ is empty. Then $P_0 - P_1 = \emptyset$, $P_0 - P_0$ can be anything $\neq \emptyset$, and $P_0 - O$ consist of any nonempty subset of $\mathcal{P}_0$ each paired with D. The total contribution for each D is $\tilde{\nu}(n-1)$.

ii. $\mathcal{I}_1$ is one half-cube. $\mathcal{I}_1$ must be either U or U', and then $P_0 - P_0$ must consist of the single element which is the complement of D in $\mathcal{I}_1$. The image of $P_0 - P_1$ in $\mathcal{P}_0$ must be one or both the half-planes of $P_0 - P_0$; similarly for the image of $P_0 - O$ in $\mathcal{P}_0$. For each D we get eighteen possibilities, contributing in all $-2\tilde{\nu}(n-1)$ for each D.

iii. $\mathcal{I}_1$ is two or more distinct half-cubes. The fact that $P_0 - P_0 \neq \emptyset$ gives an $O - O$ irrelevance.

(b) $P_0 - P_0 = \emptyset$, $P_1 - P_1 \neq \emptyset$— same as case (a).

(c) $P_0 - P_0 = \emptyset = P_1 - P_1$. Consider the number of half-cubes, m_0, m_1 of $\mathcal{P}_0, \mathcal{P}_1$ occuring among $P_0 - P_1, P_i - O$. Since D is nonempty either n_0 or n_1 is nonzero.

i. $n_0 > 0, n_1 = 0$: then $P_0 - P_1 = P_1 - O = \emptyset$, and $P_0 - O$ consists of any nonempty subset of $\mathcal{P}_0$ paired with D. The contribution for each D is $-\tilde{\nu}(n-1)$.

ii. $n_0 = 0, n_1 > 0$: the same.

iii. $n_0 = 1 = n_1$: Let the images be U_0, U_1. We consider the cases where U_0, U_1 are:

A. the same. this gives $\mathcal{O}-\mathcal{O}$ irrelevancy.

B. complementary. There are $2(n-1)$ possible U_0, U_1 pairs. For each pair we must sum over the number of ways that $P_0 - P_1, P_i - O$ can have images U_i in $\mathcal{P}_i$ and D in $\mathcal{O}$. Letting $A = \{U_0, U_1\}, B = \{U_0, D\}, C = \{U_1, D\}$, this is equivalent to summing (-1) parity of all subsets of $\{A, B, C\}$ which cover U_0, U_1, D. This gives 2, which we mulitply by each one's contribution of $\tilde{\nu}(n-1)$ and the $2(n-1)$ ways it can arise, for a total contriubtion of $4(n-1)\tilde{\nu}(n-1)$ (for each D).

C. skew. Here U_0, U_1 must be complements of U, U' in order to avoid irrelevancy. There are two ways of chosing U_0, U_1, and the same $\{A, B, C\}$ argument shows that we multiply this by 2 for a total of $4\tilde{\nu}(n-1)$ per D.

iv. $n_0 = 2, n_1 = 1$: Let U_0, U_0' and U_1 be the images in $\mathcal{P}_0$ and $\mathcal{P}_1$. There are only two possibilities to avoid irrelevance: $U_1 = U$, $U_0 = U'$, $U_0' = \overline{U}$, or the same with U, U' reversed (where $\bar{}$ denotes the complement). Each of these reduces to counting which of $\{A, B, C, D, E\}$ cover U_0, U_0', U_1, D, which A, B, C as before and $D = \{U_0', U_1\}, E = \{U_0', D\}$. This complex has $\nu = -2$, giving a total contribution of $-4\tilde{\nu}(n-1)$ for each D.

v. $n_0 = 1, n_1 = 2$: the same.

vi. $n_0, n_1 \geq 2$: this forces $\mathcal{O} - \mathcal{O}$ irrelevance.

The total contribution to $\nu(n)$ from all the above parts are:

$$(1-2n)\nu(n-1) + 4(2n-3)(n-1)(n-2)\tilde{\nu}(n-1)$$

Next we analyze $\tilde{\nu}(n) = \nu(\mathcal{B}_W^{2,n})$ for a fixed subcube W of codim 2 in $\mathbf{B}^n$. We fix such a W, which we can assume lies in $\mathcal{O}$, and viewing W as a codim 2 subcube of $\mathbf{B}^{n-1}$ we have W is the intersection of two half-cubes Q, Q' which we fix.

The same cases as before can be studied, and they yield:

1. (a) i. $-\tilde{\nu}(n-1)$.
   ii. A. 0
   B. Here the unique pair of $P_0 - P_0$ must be $\{Q, Q'\}$ to avoid irrelevance, i.e. $Q = U_0, Q' = U_0'$ or $Q = U_0', Q' = U_0$. So we get a total contribution of $2\tilde{\nu}(n-1)$.
   C. Skew. As in the previous case we get $2\tilde{\nu}(n-1)$ for similar reasons.
   iii. $-2\tilde{\nu}(n-1)$, since, again, $\{U_0, U_0'\}$ must be $\{Q, Q'\}$.
   (b) same as case (a).
   (c) i. $\tilde{\nu}(n-1)$
   ii. A. 0
   B. $-2(n-1)\nu(n)$.
   C. Here $\{U_0, U_1\}$ must be $\{Q, Q'\}$ to avoid irrelevance, yielding $-2\tilde{\nu}(n-1)$.
   iii. To avoid irrelevance $\{U_0, U_0'\}$ must be $\{Q, Q'\}$, yielding $4\tilde{\nu}(n-1)$.

2. Here D must be, of course, W. Modulo this fact everything else is unchanged.
   (a) i. $\tilde{\nu}(n-1)$.
   ii. $-2\tilde{\nu}(n-1)$
   iii. 0
   (b) same as case (a).
   (c) i. $n_0 > 0, n_1 = 0$: $-\tilde{\nu}(n-1)$.
   ii. $n_0 = 0, n_1 > 0$: the same.
   iii. $n_0 = 1 = n_1$:
   A. 0
   B. complementary. $4(n-1)\tilde{\nu}(n-1)$
   C. skew. $4\tilde{\nu}(n-1)$

iv. $n_0 = 2, n_1 = 1$: $-4\tilde{\nu}(n-1)$ for each D.

v. $n_0 = 1, n_1 = 2$: the same.

vi. $n_0, n_1 \geq 2$: 0

The total contribution to $\tilde{\nu}(n)$ from all the above parts are:

$$(2n-5)\tilde{\nu}(n-1).$$

C Details of the c_2 or $k = 3$ calculation

The only details left are to verify the recurrence for $\nu_0(n+1)$. For $(|P_0|, |P_1|)$ equal $(0,0), (0, \neq 0), (\neq 0, 0)$ the total contribution is $-\nu_0(n)$ (as is seen by the same considerations as those used to compute P_{00}). The following other $(|P_0|, |P_1|)$ cases contribute; we list the cases up to symmetry (including, in some cases P_0, P_1 swapping symmetries); below $f(m)$ denotes $f(m,3) = (-1)^n\binom{n-1}{2}$, ν_i denotes $\nu_i(n)$ and $|\nu_i|$ denotes the number of sets similar to $\mathcal{W}_i$ in $\mathbf{B}^n$ (and similarly for $\tilde{\nu}$), so $|\nu_1| = 2n4\binom{n-1}{2}$, $|\nu_3| = 2\binom{n}{2}$, $|\nu_5| = 4\binom{n}{2}4\binom{n-2}{2}/2$.

$= (1,1)$. The cases of interest are $(P_0; P_1) =$

1. $(x_1; \overline{x_1})$. For $|O| = 1$ we get $2n4\binom{n}{2}\tilde{\nu}$ times $f(3) = 1$. For $|O| = 2$ we get $2n$ times $f(4)$ times the sum of $|\nu_i|\nu_i$ for $i = 1,3,5$.
2. $(x_1; x_2)$. For $|O| = 1$ the possibilities are:
   (a) x_1x_2. This gives $2n2(n-1)\tilde{\nu}f(3)$.
   (b) $\overline{x_1}x_2$. This gives $2n2(n-1)\nu_3 f(3)$.
   (c) x_1x_3. This gives $2n2(n-1)4(n-2)\nu_1 f(3)$.
   (d) x_3x_4. This gives $2n2(n-1)4\binom{n-2}{2}\nu_5 f(3)$.

   For $|O| = 2$ we get the same ν_i terms, $i = 1,3,5$, with $f(3)$ replaced by $f(4)$ (and no other terms).

$= (2,1)$. The cases of interest are $P_0 = \{x_1, x_2\}$ (always introducing a factor of $4\binom{n}{2}$ which we omit from the following), and $P_1 =$

1. $\overline{x_1}$. For $O = \emptyset$ this yields $2f(3)\tilde{\nu}$. For $|O| = 1$ the cases are $O =$
   (a) $\overline{x_1}x_2$. Gives $2f(3)\tilde{\nu}$.
   (b) $x_1\overline{x_2}$. Gives $2f(3)\nu_3$.
   (c) x_1x_3 or $\overline{x_2}x_3$. Gives $2f(3)2 \cdot 2(n-2)\nu_1$.
   (d) x_3x_4. Gives $2f(3)4\binom{n-2}{2}\nu_5$.

   For $|O| = 2$ we get the same ν_i terms with $i = 1,3,5$ and $f(3)$ replaced by $f(4)$.
2. x_3. Here O must be a subset of $\{x_1x_3, x_2x_3\}$, for a total of $2(n-2)\nu_1$ times $f(3) + 2f(4) + f(5) = -1$.

$= (2,1)$. This is the same as $= (1,2)$.

$= (3,1)$. Up to symmetry the only case of interest is $P_1 = \{x_1, x_2, x_3\}$ and $P_2 = \overline{x_1}$. The choice of P_1 gives a factor of $8\binom{n}{3}$, that of P_2 gives a factor of 3 (we can have $P_2 = \overline{x_i}$ for $i = 1,2,3$). Then O must be a subset of $\{\overline{x_1}x_2, \overline{x_1}x_3\}$, for a total contribution of $f(4) + 2f(5) + f(6) = 1$ times ν_1 time the previous factors, $8\binom{n}{3}3$.

$= (3,1)$. This is the same as $= (1,3)$.

$= (2,2)$. Up to symmetry we must have $P_1 = \{x_1, x_2\}$, $P_2 = \{\overline{x_1}, \overline{x_2}\}$, and O a subset of $\{x_1\overline{x_2}, \overline{x_1}x_2\}$, giving a total contribution of $4\binom{n}{2}\nu_3(f(4) + 2f(5) + f(6))$.

The total of all these terms yields the formula stated previously for $\nu_0(n+1)$.

References

[Bjö84] Anders Björner. Some combinatorial and algebraic properties of coxeter complexes and tits buildings. *Advances in Mathematics*, 52:173–212, 1984.

[BT82] R. Bott and L. Tu. *Differential Forms in Algebraic Topology*. Springer-Verlag, 1982.

[CF90] Ming-Te Chao and John Franco. Probabilistic Analysis of a Generalization of the Unit-Clause Literal Selection Heuristics for the k Satisfiability Problem. *Information Sciences*, 51:289–314, 1990.

[CR92] V. Chvátal and B. Reed. Mick gets his (the odds are on his side) [sic]. preprint, 1992.

[DC91] Bruce R. Donald and David R. Chang. On computing the homology type of a triangulation. In *32nd Annual Symposium on Foundations of Computer Science*, pages 650–661, 1991. also preprint (of the full version).

[dlV] W. Fernandez de la Vega. personal communication.

[dlV92] W. Fernandez de la Vega. On random 2-SAT. preprint, 1992.

[Fri91] J. Friedman. On the second eigenvalue and random walks in random d-regular graphs. *Combinatorica*, 11:331–362, 1991.

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1978.

[Knu73] Donald E. Knuth. *The Art of Computer Programming*, volume 1: Fundamental Algorithms. Addison-Wesley, Reading, Massachusetts, second edition, 1973.

[KSS84] J. Kahn, M. Saks, and D. Sturtevant. A topological approach to evasiveness. *Combinatorica*, 4:297–306, 1984.

[Mun84] James R. Munkrees. *Elements of Algebraic Topology*. Benjamin/Cummings, 1984.

[RS82] C.P. Rourke and B.J. Sanderson. *Introduction to Piecewise-Linear Topology*. Springer-Verlag, 1982.

[SB80] J. Stoer and R. Bullirsch. *Introduction to Numerical Analysis.* Springer-Verlag, 1980.

[Val84] L.G. Valiant. Short monotone formulae for the majority function. *Journal of Algorithms*, 5:363–366, 1984.

[Wel80] R.O. Wells. *Differential Analysis on Complex Manifolds.* Springer-Verlag, 1980.