A NOTE ON POSET GEOMETRIES

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Abstract

In this note we describe how varying the geometric representation of a poset can be applied to "poset balancing." We show that the 1/3, 2/3 balancing property holds for a certain class of posets whose number of relations is sufficiently small, in a certain sense.

1 Introduction

Given a poset (partially ordered set), for elements x and y let $p(x<y)$ denote the fraction of completions of the partial order to a total order in which $x<y$. In [Fre76], M. Fredman, conjectured (and N. Linial in [Lin84]) that any non-total poset (i.e. poset which is not a total order) has two elements, x, y, for which $1/3 \leq p(x < y) \leq 2/3$. This conjecture arose in the context of studying the information theoretic bound of the complexity of sorting the elements of the poset. The non-trivial non-total three element poset shows this conjecture to be as optimistic as possible.

To date the conjecture is unresolved, but using convexity in geometric realizations of the posets, such as the techniques of Stanley in [Sta81], theorems have been proven with the $1/3, 2/3$ replaced by different constants. In [KS84], J. Kahn and M. Saks, proved the above conjecture with $3/11, 8/11$ as constants. In [KL88], J. Kahn and N. Linial gave a simpler proof of the conjecture with $1/(2e), (2e-1)/(2e)$ as constants.

Both proofs are based on convex geometry involving a geometric realization of S_n, the group of permutations on n objects. The point of this note is to remark that by varying the geometries one can sometimes get better results. This is definitely

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true when the poset has few enough relations, in a certain sense. The geometries we use are suggested by the standard realization of S_n as its associated Coxeter complex (see, for example, [Ron89]). We combine the varying geometries with the simplified technique of [KL88] to obtain improved results such posets. In this note we prove:

Theorem 1.1 *For any $\epsilon > 0$ there is a C such that the following is true. Let P be a poset on $\{x_1, \ldots, x_n\}$, and let a_i and b_i denote the number of elements respectively $>$ and $<$ than x_i in P. If for every permutation $\sigma \in S_n$ we have*

$$\sum_{i=1}^{n} \left(\frac{\sigma(i)}{a_i + 1} + \frac{n + 1 - \sigma(i)}{b_i + 1} \right) \geq C,$$

then P has elements x, y with $(1/e) - \epsilon \leq p(x < y) \leq 1/2$.

This is the precise sense of P having "few enough relations" mentioned earlier. As applications we have:

Theorem 1.2 *For any $\epsilon > 0$ there is a C such that if P either (1) has at least $C\sqrt{n}$ maximal (or minimal) elements, or (2) has no chain of length $\geq C + 2\log_2 \log n$, then P has elements x, y with $(1/e) - \epsilon \leq p(x < y) \leq 1/2$. For any $\epsilon > 0$ and $\nu > 0$ there is a $\mu > 0$ such that the same conclusion holds if P has some νn of its elements each unrelated to at least $(1 - \mu)n$ other elements.*

We also note that other special cases of the poset $(1/3, 2/3)$ conjecture have been resolved. In [Lin84], N. Linial proves the conjecture for posets of width 2. In [Kom], J. Komlos proves that for any $\epsilon > 0$ there is a function $f_\epsilon(n) = o(n)$ such that any poset with at least $f_\epsilon(n)$ minimal elements has two elements x, y such that $|p(x < y) - 1/2| \leq \epsilon$; here $f(n)/n$ decreases to zero exponentially fast in the inverse of some Ramsey-type function of n. Also J. Kahn and M. Saks have conjectured that as the width of the poset tends to infinity, a $|p(x < y) - 1/2| \leq o(1)$ balancing result should hold.

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2 Variants of a Standard Realization

A standard geometric realization of S_n is its associated Coxeter complex (see [Ron89]). One views this as a triangulation of the $n-2$-dimensional sphere. The convex polytope determined as the convex hull of the simplices of this triangulation (this polytope looks like a beachball) is a realization of S_n such that the realization of every poset is convex. Of course, there is no reason to insist that this polytope's vertices all lie on one sphere. By moving certain vertices further or closer to the center, we get different convex polytopes.

So consider n points $v_1, \ldots, v_n \in \mathbf{R}^{n-1}$ not contained in any hyperplane of dimension $n-2$. Every point $v \in \mathbf{R}^{n-1}$ can be uniquely written as

$$v = \sum \alpha_i v_i, \qquad \text{with} \quad \sum \alpha_i = 1. \tag{2.1}$$

For a permutation of $\{1, \ldots, n\}$, $\sigma = \{i_1, \ldots, i_n\}$, let

$$A_\sigma = \left\{ \sum \alpha_i v_i \mid \alpha_{i_1} \geq \cdots \geq \alpha_{i_n} \right\}.$$

If U is any convex body, say, containing the v_i, then $\sigma \mapsto U_\sigma \equiv A_\sigma \cap U$ is a realization of S_n in which every poset corresponds to a convex subset of U. U_σ will be *adjacent* to U_τ, i.e. will share a facet (i.e. an $(n-2)$ dimensional face), iff σ and τ differ by some transposition (i,j), and in that case the facet lies on the hyperplanes H_{ij} containing $(v_i + v_j)/2$ and all v_k with $k \neq i,j$. When no confusion will occur, we will often simply refer to this facet as H_{ij}.

For future reference, the α_i's in equation 2.1 are called the *barycentric coordinates* of v (with respect to the v_i's). If U is the simplex spanned by the v_i's, the *barycentric distance* of v to a facet, F, of U (i.e. a simplex spanned by any $n-1$ distinct v_i's), is the barycentric coordinate of v with respect to the v_i not contained in F.

We now make some explicit calculations to describe various of choices of U. For simplicity, we perform them in $\mathbf{R}^n$ restricted to the hyperplane $x_1 + \cdots + x_n = 1$. For a subset $S \subset \{1, \ldots, n\}$ let e_S be the vector which is 1 on the i-th coordinate if $i \in S$ and 0 elsewhere. Let $Q = (1/n, \ldots, 1/n)$. For positive real $\theta_1, \ldots, \theta_n$ consider the collection of points

$$v_S = \theta_S e_S + \Big(1 - |S|\theta_S\Big) Q$$

with $\theta_S = \theta_{|S|}$, ranging over all non-empty proper subsets, S, of $\{1, \ldots, n\}$. Clearly all these vertices lie on the hyperplane $x_1 + \cdots + x_n = 1$. Let U be their convex hull.

Lemma 2.1 *The following two conditions are equivalent: (1) no v_S is in the interior of U, and (2) for all $i < j$*

$$i\theta_i \leq j\theta_j \qquad \textit{and} \qquad (n-i)\theta_i \geq (n-j)\theta_j. \tag{2.2}$$

Proof By symmetry the first condition is equivalent to saying that for any $|S|$, the centers of mass of the sets

$$E_{j,k} = \Big\{ e_T \;\Big|\; |T \cap S| = j, |T - S| = k \Big\}$$

for all k, j lie between (or on) Q and e_S. Each of these gives an inequality between $\theta_{|S|}$ and θ_{j+k} which is exactly of the form of those of equation 2.2, (except when $j + k = |S|$ which is trivial), and conversely each inequality in equation 2.2 arises in this way.

□

For future reference we note that the distance of v_S to a halfplane H_{ij} is just $\theta_S/\sqrt{2}$ if exactly one of i, j are in S (and 0 otherwise); this is seen by noting that the reflection through H_{ij} merely exchanges the i-th and j-th coordinates. We also note some familiar choices of θ_i. The choice $\theta_i = 1/i$ and $\theta_i = 1/(n-i)$ are simplices with vertex sets $\{v_S\}$ ranging over S of respective sizes 1 and $n-1$. Choosing the v_S's to be equidistant from P gives

$$\theta_i = \sqrt{\frac{n}{i(n-i)}}.$$

We now describe some features of varying the geometry. First we make the observation that in any poset there exists an ordering of the elements $\sigma = \{x_1, \cdots, x_n\}$ such that $p(x_i > x_{i+1}) \geq 1/2$ for all i. We call such a σ *optimal.* Its existence follows from the fact that any tournament has a Hamiltonian path. This statement also implies that $p(x_i > x_j) > 2/3$ for all $i > j$ if P is a counterexample to Fredman's conjecture. The point to our method is that fixing P and such an ordering we can choose the geometry best suited to the situation at hand. We will apply this to the centroid method used in [KS84] and [KL88], using the simplified technique of the latter. We explain this in the next section.

It is sometimes easier to visualize the problem, and amusing, if not particularly useful, to state the poset problem in the "real-estate" terminology (see [Ron89]). Given a Coxeter complex, there are two natural notions of convexity for a subset of chambers— that of metric convexity, and that of being an intersection of half-apartments. It is easy to see that these two notions are equivalent; in the case of S_n a convex set is merely a poset, and our question is to try to find a wall which divides a given convex set P into roughly equal parts. For example, since any collection of chambers is non-trivially divided by some wall, it follows by descending induction that there is a collection of size 1 all of whose bounding half-apartments contain more than half of P; this proves the existence of an optimal σ.

3 Centroid Type Arguments

We review the techniques in [KL88]. They start with the observation:

Lemma 3.1 *Let C be a convex body in $\mathbf{R}^m$ such that the centroid of C has x_1 coordinate $-\alpha$, and contains points with x_1 coordinates u and $-w$, with some $u, w, \alpha \geq 0$. Then*

$$\frac{|C \cup \{x_1 \geq 0\}|}{|C|} \geq \min_{U \geq u, W \geq w} \left(\frac{1}{1 + \frac{1}{m-1} + \frac{(m+1)\alpha - W}{(m-1)U}} \right)^{m-1} \frac{1}{1 + \frac{W}{U}}. \tag{3.1}$$

Furthermore the right-hand-side above is minimized at $U = u, W = w$ if $u \geq u^*$ and otherwise at $U = u^*, W = w$, where

$$u^* = \frac{wm\big(w - \alpha(m+1)\big)}{w + \alpha(m^2 - 1)}.$$

Proof The proof is a simple argument which shows that the worst case C is a "double cone," in the spirit of B.S. Mityagin (see [Mit68]). Calculating the worst case volume ratio on this basis is easy, and yields equation 3.1, which is essentially straight from [KL88]. Differentiating in U and W yields the second part, using the fact that $w \geq (\alpha(m+1) + u)/m$ always holds in the above situation.

□

If $\alpha = 0$ in the above, the above volume ratio is at least $1/e$, which is Mityagin's result. So we can expect volume ratios close to this if α is small enough.

The argument in [KL88] is as follows. Fix a realization of S_n as in the previous section. Let σ be an optimal total order, and let c be the centroid of the poset P, where we identify P with its realization. We can assume $c \in A_\sigma$ (or else we can apply Mityagin's result), so consider c's barycentric coordinates with respect to the vertices of A_σ. If H_{ij} is a facet of A_σ with i and j related in P (i.e. either $i < j$ or $i > j$ in the partial ordering), then P itself lies to one side of H_{ij} and it easily follows that the barycentric distance of c to H_{ij} is at least $1/n$. Hence there must exist some facet, H_{ij}, of A_σ whose barycentric distance is $\leq 1/n$ such that i, j are unrelated in P, and in particular $A_{\sigma'}$ lies in P where σ' is σ followed by the transposition (i, j). Then we can apply lemma 3.1 with $m = n-1, u = w = 1, \alpha = 1/(m+1)$ which gives a volume ratio $\geq 1/(2e)$.

Actually, the realization used in [KL88], [KS84], and [Sta81] is different from ours; namely, they use the cube $[0,1]^n$ with A_σ being the set $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$. In this realization, when H_{ij} is a facet of A_σ, there is never any point in P which is further away from H_{ij} than one of the vertices of A_σ. If we use our realizations, then it can happen that some of P's points are futher away, and we can get better results. This can be guarenteed to be the case when P is "sufficiently sparse."

More precisely, recall from the last section that the distance from a vertex v_S to is proportional to θ_S. The choice of $\theta_i = 1$ would yield a situation like the cube realization, but varying the θ_i's allows some improvement. Varying the θ_i's involves slightly different applications of lemma 3.1, namely:

Lemma 3.2 *For any positive ϵ there is a positive $\delta = \delta(\epsilon)$ such that if, in lemma 3.1, α is less than $\delta u/m$ or $\delta w/m$ in lemma 3.1, then*

$$\frac{|C \cup \{x_1 \geq 0\}|}{|C|} \geq \frac{1}{e} - \epsilon.$$

Proof This is an easy calculation. If we have $(m+1)\alpha/U \leq \delta$ for some small δ (slightly different than the δ in the statement of the lemma), then the right-hand-side of equation 3.1 with $U = u$ is bounded below by

$$\left(\frac{1}{1+\frac{1+\delta}{m-1}+\frac{-W}{(m-1)u}}\right)^{m-1} \frac{1}{1+\frac{W}{u}}.$$

By substituting $t = W/u$ and differentiating, we can see that for any fixed m there is a δ making the above expression $\geq (1/e) - \epsilon$ for all positive W. On the other hand, for large m the above expression is

$$\geq \left(1+O\left(\frac{1}{m}\right)\right) e^{-(1+\delta)} e^{W/u} \frac{1}{1+\frac{W}{u}},$$

and as before we see that this is $\geq (1/e) - \epsilon$ for m sufficiently large and some positive δ (independent of m). This proves the first case of the lemma. In the second case we write $(m+1)\alpha - W \leq -(1-\delta)W$ and proceed similarly, showing that a small enough δ yields the desired lower bound.

□

Returning to the situation at hand, given a poset P fix settings of the θ_i's, consider any optimal $\sigma = (x_1, \ldots, x_n)$, and let c be the centroid of P. Being interested in bounds of the form $1/e, 1-(1/e)$ or worse, we can assume that c lies in A_σ. The vertices of A_σ are v_{S_i} with $S_i = \{x_1, \ldots, x_i\}$, $1 \leq i \leq n-1$. Let

$$c = \sum_{i=1}^{n-1} v_{S_i} \alpha_{S_i}$$

be the barycentric representation of c in A_σ. For each i fix a set T_i which contains exactly one of x_i, x_{i+1} and with $v_{T_i} \in P$; usually we'll take T_i to be the smallest or largest such set, depending on the choice of θ_i's. T_i is any set whose elements are not $<$ any element not in T_i (in the partial order P).

Corollary 3.3 *If $x_i > x_{i+1}$ in P, then*

$$\alpha_{S_i} \geq \frac{1}{n-1} \frac{\theta_{T_i}}{\theta_{S_i}}.$$

If not, then

$$\alpha_{S_i} \geq \delta(\epsilon) \frac{1}{n-1} \frac{\theta_{T_i}}{\theta_{S_i}},$$

unless $p(x_i < x_{i+1}) \geq (1/e) - \epsilon$, for any ϵ with δ as in lemma 3.2.

Proof In the first case, the facet, H, of A_σ opposite v_{S_i}, bounds P, and yet there is a point of P, v_{T_i}, whose distance to H is $\theta_{T_i}/\sqrt{2}$. c's distance to H must be at least $1/(n-1)$ of T_i's distance to H. On the other hand, c's distance to H is precisely $\alpha_{S_i}\theta_{S_i}/\sqrt{2}$.

The second case follows from lemma 3.2, with similar distance considerations.

□

Corollary 3.4 *If*

$$\sum_{i=1}^{n-1} \frac{\theta_{T_i}}{\theta_{S_i}} \geq \frac{n-1}{\delta(\epsilon)}, \tag{3.2}$$

then there exists an i with $p(x_i < x_{i+1}) \geq (1/e) - \epsilon$.

We now seek situations in which we can guarentee equation 3.2 will hold for appropriate θ_i's.

Theorem 3.5 *For any $\epsilon > 0$ there is a C such that if P contains at least $C\sqrt{n}$ maximal elements, then there exists elements x, y*

$$\frac{1}{e} - \epsilon \leq p(x < y) \leq \frac{1}{2}.$$

Proof Take $\theta_i = 1/i$ in the above. If, in the above circumstances, x_i is maximal, then taking $T_i = \{x_i\}$ gives $\theta_{T_i}/\theta_{S_i} = |S_i|$. Hence

$$\sum_{i=1}^{n-1} \frac{\theta_{T_i}}{\theta_{S_i}} \geq \sum_{i=1}^{C\sqrt{n}} i \geq \frac{n-1}{\delta}$$

for sufficiently large C.

□

More generally we have

Theorem 3.6 *For any $\epsilon > 0$ there is a C such that the following is true. Let P be a poset on $\{x_1, \ldots, x_n\}$, and let a_i and b_i denote the number of elements respectively $>$ and $<$ than x_i in P. If for every permutation $\sigma \in S_n$ we have*

$$\sum_{i=1}^{n} \left(\frac{\sigma(i)}{a_i + 1} + \frac{n+1-\sigma(i)}{b_i + 1} \right) \geq C, \tag{3.3}$$

then P has elements x, y with $(1/e) - \epsilon \leq p(x < y) \leq 1/2$.

Proof Consider the two choices for θ_i, $1/i$ and $1/(n-i)$. Taking C to be $2/\delta(\epsilon)$, we find that if equation 3.3 holds then we can apply corollary 3.4 for one of these two choices of θ_i.

□

While the condition in theorem 3.6 requires optimizing over σ and is not entirely explicit, in many cases it is not hard to check that it holds, such as when there exist $C\sqrt{n}$ maximal elements. We give some other examples to which theorem 3.6 can be applied.

Theorem 3.7 *For any $\epsilon > 0$ the condition (and therefore conclusion) of theorem 3.6 holds if every chain in P has length $\leq C'(\epsilon) + 2\log_2 \log n$.*

Theorem 3.8 *For any $\epsilon, \nu > 0$ there is a $\mu > 0$ such that the condition (and therefore conclusion) of theorem 3.6 holds if some νn of P's elements are each unrelated to more than $(1-\mu)n$ (possibly different) elements of P.*

Proof For the latter theorem, each element x_i unrelated to more than $(1-\mu)n$ has both a_i and b_i less than μn. Hence it suffices to chose μ so that ν/μ exceeds $C(\epsilon)$ of theorem 3.6. To prove the former, let X_i be the number of nodes whose longest chain from a maximal element is of length $i+1$; for example, X_1 is the set of maximal elements. Then any two elements of any X_i are unrelated. Then

$$\sum_{i=1}^{n} \frac{\sigma(i)}{a_1+1} \geq \sum_{i=1}^{n_1} i + \sum_{i=n_1+1}^{n_2} \frac{i}{n_1+1} + \sum_{i=n_1+n_2+1}^{n_3} \frac{i}{n_1+n_2+1} + \cdots$$

which, within a constant, is

$$\geq n_1^2 + \frac{n_2^2}{n_1} + \frac{n_3^2}{n_2} + \cdots. \tag{3.4}$$

A similar estimate holds for the sum in equation 3.3 involving the b_i's. Now let k be the length of the longest chain in P. If the condition for theorem 3.6 is not met, then the expression in equation 3.4 must be bounded by Cn for some constant C. Then we conclude $n_1 \leq \sqrt{Cn}$, and then $n_2 \leq (Cn)^{3/4}$, and more generally

$$n_j \leq (Cn)^{1-\frac{1}{2^j}}.$$

Now let k be the length of the longest chain in P. Applying the same argument to the sum in equation 3.3 involving the b_i's we conclude that

$$n_{k-j} \leq (Cn)^{1-\frac{1}{2^j}}$$

for all j. Hence,

$$n = n_1 + \cdots + n_k \leq k(Cn)^{1-\frac{1}{2^{(k+1)/2}}},$$

and thus $(k+1)/2 \geq \log_2 \log n + C'$. Hence if $k \leq C'' + 2\log_2 \log n$ theorem 3.6 must apply.

□

4 Concluding Remarks

There are some other possible variants on these techniques. For one thing, we can vary the θ_S's even over S's of the same size. Of course, it may no longer be true that the A_σ's all have the same volume, but if we are only interested in $1/3, 2/3$ type results we might have some room for slight variations of volume.

On some level it seems appealing to phrase the poset question in terms of finding a wall separating a convex set of chambers in a Coxeter complex into roughly equal sizes, but it is not clear if this is of any use. It is easy to see that any convex set of a general Coxeter complex on k generators has a wall separating it into sets of fractional sizes between $1/(k+1), k/(k+1)$, and that this is the best one can say. From this point of view it is clear that for the poset problem one is making use of the special fact that most of the generators of S_n commute.

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