GRAPH DECOMPOSITIONS WITH APPLICATIONS TO CIRCLE AND PERMUTATION GRAPHS

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Abstract

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Just as integers can be decomposed into prime factors, it is possible to define primeness on graphs so that graphs may also be decomposed into their prime components. Under the definitions presented in this work, several non-isomorphic graphs may yield identical factorings. An efficient implementation of an algorithm which Cunningham introduced to decompose graphs is shown. A second algorithm to decompose graphs is also shown. This second algorithm is shown because it is also used in the inductive proof of a property of prime graphs - each vertex in a prime graph must be contained in a certain induced subgraph of the prime graph. A close link between prime graphs and certain classes of intersection graphs, namely circle graphs and permutation graphs, is established, and a recognition algorithm is presented for each.

Acknowledgements

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I dedicate this thesis to my parents, Ferenc and Reka Gabor, who have always stood behind me and given me their love and guidance.

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1. Introduction and Definitions

1.1. Introduction

An area of graph theory that has received substantial interest in the past thirty years is the area of intersection graphs. Intersection graphs are those graphs whose vertices are some objects, and which vertices are adjacent iff the objects they are intersect. Common examples of objects include intervals on a number line, chords on a circle, and rectangles in the plane.

Intersection graphs may often be associated in a natural way with problems arising in engineering and the sciences. For example, interval graphs (graphs whose vertices correspond to intervals on the number line) have been used in modelling a problem in genetics on gene structure called Benzer's problem ([Be]), and a scheduling problem from [Ha]. There are many other areas in which interval graphs arise; [Go], [Tr], and [Ro] provide extensive surveys in this area. Permutation graphs (intersection graphs where the vertices are line segments having one endpoint on each of two given parallel lines) arise in [L] in a VLSI problem involving routing of nets and in [Kn] in the problem of optimal scheduling for reallocations of memory space in a computer.

Other areas of applications include solving certain NP-complete problems in polynomial time on restricted sets of graphs. For example, the maximum weighted independent set problem and the maximum weighted clique problem can be solved in polynomial time on circle graphs (intersection graphs where the objects are chords on a circle) ([Ga], [Hs]).

Graph decomposition is an area of graph theory which has been extant for about twenty years. Early references are listed in [Cu], which is one of the main works on the topic. Graph factoring is analogous to the factoring of integers; in both cases they factor uniquely into their prime components. However, while graphs may be decomposed uniquely, it is also the case that non-isomorphic graphs may yield identical factorings. The motivation behind factoring a graph is that it may be easier to solve a given problem on smaller, factored pieces and then assemble a solution to the original problem, as with the optimal stable set problem in [Ch] and sequencing problems in [Si].

This thesis explores further the concept of factoring undirected graphs and applying factoring to a specific problem: recognition of circle and permutation graphs. One important result is a new algorithm by which to factor graphs. By factoring one may get to the heart of several problems: many special cases are removed (as with the recognition problems). It is hoped that by solving this important problem, a suitable framework has been provided for application to further problems (such as in [Ch] and [Si]).

The second important result is a characterization of unfactorable graphs. An initial characterization of unfactorable graphs is that each one must contain an induced subgraph isomorphic to some member of a certain set of graphs. This delineation is then strengthened considerably to say that each vertex of an unfactorable graph is contained in an induced subgraph isomorphic to some member of a certain set of graphs. Although subgraph theorems are not uncommon (e.g. [Tu], [BM], [GiH]), statements about each vertex in a class of graphs are. An example of this type of characterization is that each vertex in a graph with a Hamiltonian circuit has even degree.

The third result is the application of graph decomposition to the recognition of circle graphs and permutation graphs. Circle graph recognition was posed as an open problem in [Go] and [GJ]. [Bu] conjectured that the problem was NP-complete. Later, several distinct polynomial time methods were discovered for recognition of circle graphs in [B1],[B2],[GH], and [GH2]. [B1] was primarily a proof that recognition was in P, while [B2] included an $O(|V|^4)$ algorithm. The algorithms in [GH] and [GH2] are each $O(|V| \times |E|)$. The former is quite complicated while the latter is an outline of the version in this document. The version offered here parallels the graph decomposition algorithm while previous versions have not.

[SP3] has claimed an $O(|V|^2)$ recognition algorithm for prime circle graphs. One of the two major remaining questions is whether there is a faster decomposition algorithm for arbitrary graphs. The other major remaining problem is to develop a forbidden subgraph theorem for circle graphs and link it to the concept of local complementation developed in [B2].

1.2. Overview

In [GH] a polynomial-time algorithm is presented for the recognition of circle graphs, a class of graphs where the vertices are represented by chords on a circle and vertices are adjacent iff their corresponding chords intersect. The paper makes use of a type of factoring of graphs which we call 4–*factoring*, and demonstrates that circle graphs can be represented in essentially one way if they are not further 4–*factorable*; that is, if they are 4–*prime*. This work develops more fully the notion of 4-prime (and the related notions of 3–*prime* and 3–*factoring* which have a correspondence with permutation graphs). It is shown that not only must 4-prime and 3-prime graphs have a certain structure (Theorems 1 and 2), but that each vertex in such a graph must be contained within a certain induced subgraph (Theorems 3 and 4). The 4-factoring of graphs is also explored, leading to a polynomial-time algorithm (Lemma 1 inplemented via algorithm B1 and C1) which is also used in proving the aforementioned structural theorems, and in proving (Theorem 5 and) correctness of the recognition algorithm (Theorem 6) for circle graphs ([GH] present only an outline).

In this chapter, we define the two types of decomposition metioned above (and hence two notions of primeness) on graphs. In each case, a graph is prime if there are no two subgraphs into which it can be decomposed, and every graph is either prime or can be decomposed into a unique set of prime graphs.

Although every graph decomposes uniquely, there are distinct graphs which yield identical decompositions. Nevertheless, these compositions do have some interesting properties, impose a certain structure on both 4-prime and 3-prime graphs, and facilitate the solution of certain graph problems. Chapter 2 will examine these properties in depth. Central to Chapter 2 is a polynomial-time algorithm (Central Lemma) that decomposes a graph into its 4-prime components. This algorithm or a variant (Lemmas 2 and 3, Cor. 1) is used repeatedly throughout the rest of this dissertation both in proving various properties (Theorems 3, 4, and 5) and in solving other graph problems (Theorem 6). In chapter 2 it is used to establish the structural theorems (3 and 4) mentioned above.

Chapter 3 deals with the application of Chapter 2 to circle graphs, which are a particular class of graphs -- a type of intersection graph. Since circle graphs form a proper subset of all graphs, it is not surprising to find that there are some problems which are more easily solved on the class of circle graphs than on graphs in general. Until recently, however, the problem of recognizing whether a graph is a circle graph (and finding its appropriate representation in that event) was open. The central topic of chapter 3 is a polynomial-time algorithm for recognizing circle graphs (and finding the representation when it exists). The proof of the correctness of the algorithm relies heavily on the fact (also proved) that a 4-prime circle graph can be represented in essentially only one way (Theorem 5).

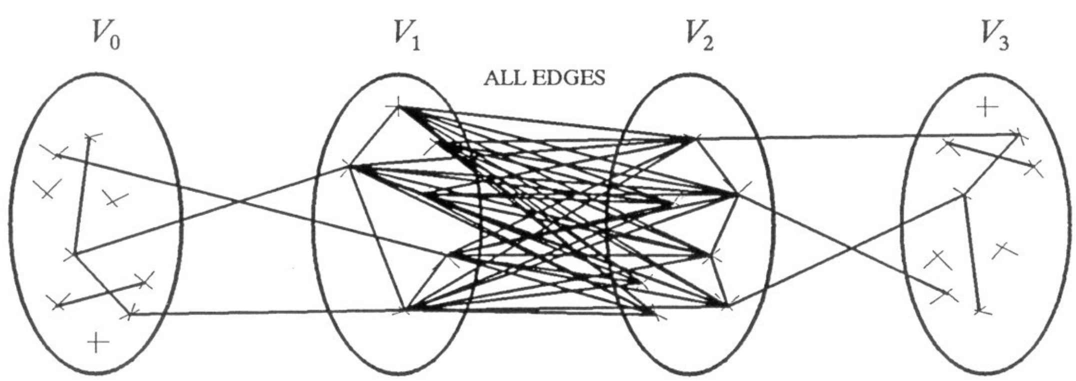
Chapter 4 concerns the application of Chapter 2 to permutation graphs. Permutation graphs form a proper subset of circle graphs, and it is possible to solve still more problems (in polynomial-time) on them than on circle graphs. A permutation graph is a circle graph where the circle on which the graph is represented may be partitioned into two arcs so that each chord has an endpoint in both arcs. The central topic of Chapter 4 is a polynomial-time algorithm for recognition of permutation graphs and for establishing their representations in that event. An important result is that permutation graphs are uniquely representable iff they are 3-prime. The results in chapter 4 are very closely tied to those of chapter 3.

1.3. Definitions

This subsection contains the conventions and definitions which will be used for the remainder of this work. The definitions of circle graphs, permutation graphs, and related terms will be found at the beginning of chapters 3 and 4, respectively.

All graphs $G = (V, E)$ considered are connected, undirected, and contain no self loops. Let $U, W \subseteq V$ and v and w be distinct vertices in V. The notation G/W will be used to mean the graph $(W, E \cap (W \times W))$ which is also referred to as the subgraph of G induced by W. $N_W(U)$ is the set of all vertices in W which are adjacent to some element of U. For ease of notation we write $N_W(w)$ in place of $N_W(\{w\})$. Vertices v and w are said to be *W–similar* whenever $N_W(v) - \{w\} = N_W(w) - \{v\}$ (i.e. their W-restricted neighborhoods excluding $\{v, w\}$ are equal). If v and w are *V–similar*, we simply say v and w are *similar* or that they are a *similar pair*. We define $dist(v, w)$ as the number of edges on the shortest path from v to w.

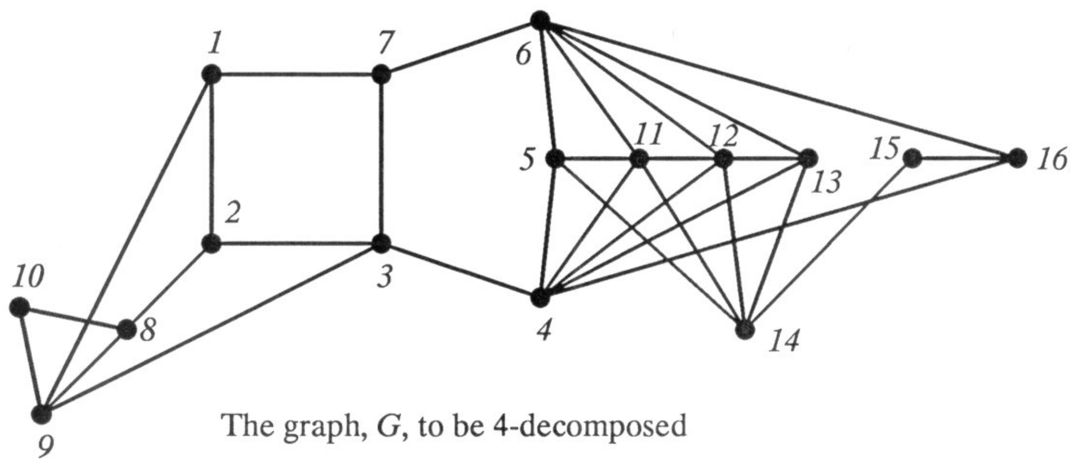
Two types of composition are defined: we say that a partition of V into four sets V_0, V_1, V_2, and V_3 is a *4–decomposing* partition if no element of V_0 is adjacent to an element of $V_2 \cup V_3$, no element of V_1 is adjacent to an element of V_3, and each element of V_1 is adjacent to each element of V_2 (see Fig. 1.1). If, in addition, $|V_0 \cup V_1| \geq 2$ and $|V_2 \cup V_3| \geq 2$, then G is *4–decomposable* or *4–factorable*. In this event, let $v_1 \in V_1$ and $v_2 \in V_2$. We say that the partition *4–factors* G into $G_1 = G/(V_0 \cup V_1 \cup \{v_2\})$ and $G_2 = G/(V_3 \cup V_2 \cup \{v_1\})$. Alternately, we say that G_1 and G_2 are a *4–factoring* for G. An induced subgraph of G, G/V, is a *4–factor*



A *4-factoring* partition for a graph

Fig 1.1

of G if G is 4-factorable and there is a graph H such that G/V and H are a 4-factoring for G. The 4–*decomposition* of a graph is defined recursively: if the graph is not 4-factorable, then its 4-decomposition is itself. If the graph is 4-factorable into G_1 and G_2, then its 4-decomposition is the 4-decomposition of G_1 and the 4-decomposition of G_2. Figure 1.2 shows the 4-decomposi-



The graph, G, to be 4-decomposed

G factors into
$G/\{1,2,3,4,5,6,7,8,9,10\}$
and $G/\{4,5,11,12,13,14,15,16\}$

$G/\{1,2,3,4,5,6,7,8,9,10\}$ factors into
$G/\{1,2,3,4,5,6,7\}$
and $G/\{2,3,8,9,10\}$

$G/\{4,5,11,12,13,14,15,16\}$ factors into
$G/\{4,5,11,12,13\}$
and $G/\{4,5,14,15,16\}$

G factors into
$G/\{1,2,3,4,5,6,7,8,9,10,14,15,16\}$
and $G/\{4,5,11,12,13\}$

$G/\{1,2,3,4,5,6,7,8,9,10,14,15,16\}$ factors into
$G/\{1,2,8,9,10\}$
and $G/\{1,2,3,4,5,6,7,14,15,16\}$

$G/\{1,2,3,4,5,6,7,14,15,16\}$ factors into
$G/\{1,2,3,4,5,6,7\}$
and $G/\{5,6,14,15,16\}$

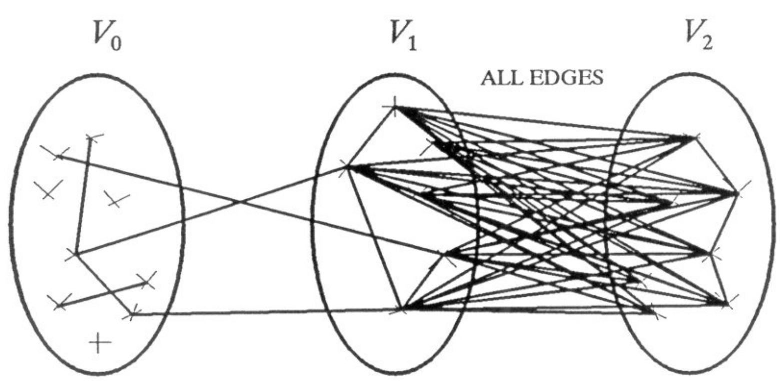
Thus, the 4-decomposition of G is into:
$G/\{1,2,3,4,5,6,7\}$,
$G/\{1,2,8,9,10\}$ (which is isomorphic to $G/\{2,3,8,9,10\}$),
$G/\{4,5,11,12,13\}$,
and $G/\{5,6,14,15,16\}$

Fig. 1.2

4-decomposition of a graph

tion of a particular graph. [Cu] shows that the 4-decomposition of a graph is unique, up to isomorphisms. G is 4–*prime* if G is not 4-decomposable and $|V| \geq 4$. Note that if G is 4-prime, then G contains no articulation points and no similar pairs.

We say that a partition of V into three sets V_0, V_1, and V_2 is a 3–*decomposing* partition if no element of V_0 is adjacent to an element of V_2, and each element of V_1 is adjacent to each element of V_2 (see Fig. 1.3). If, in addition, $|V_0 \cup V_1| \geq 1$ and $|V_2| \geq 2$, then G is 3–*decomposable* or 3–*factorable*. In this event, let $v_1 \in V_1$ and $v_2 \in V_2$. We say that the partition 3–*factors* G into $G_1 = G/(V_0 \cup V_1 \cup \{v_2\})$ and $G_2 = G/(V_2 \cup \{v_1\})$. Alternately, we say that G_1 and G_2 are a 3–*factoring* for G. An induced subgraph of G, G/V, is a 3–*factor* of G if G is 3-factorable and there is a graph H such that G/V and H are a 3-factoring for G. The 3–*decomposition* of a graph is defined recursively: if the graph is not 3-factorable, then its 3-decomposition is itself. If the graph is 3-factorable into G_1 and G_2, then its 3-decomposition is the 3-decomposition of G_1 and the 3-decomposition of G_2. G is 3–*prime* if G is not 3-decomposable and $|V| \geq 3$. It is easily verifiable that G is 3-prime iff the complement of G is 3-prime. If G is 3-prime, then G contains no similar pairs.



A *3-factoring* partition for a graph

Fig 1.3

The definitions of several important graphs follow:

(1) $P_n(p_1, p_2, \cdots, p_n)$ is the graph $(\{p_1, p_2, \cdots, p_n\}, \{(p_1, p_2), (p_2, p_3), \cdots, (p_{n-1}, p_n)\})$ (see Fig. 1.4 for example).

(2) $DW(a, b, c, d, v)$ is the graph $(\{a, b, c, d, v\}, \{(a, b), (b, c), (c, d), (v, b), (v, c)\})$ (Fig. 1.4).

P_n (a, b, c, d)

A P_n is any graph isomorphic to P_n (1,2, ... ,n)

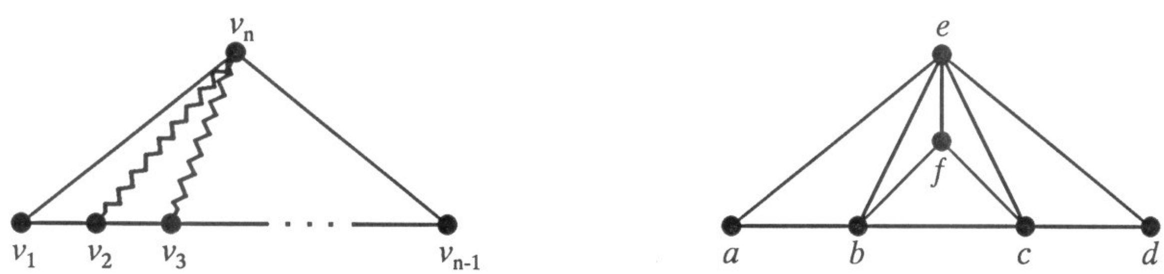
DW (a, b, c, d, e)

A DW is any graph isomorphic to DW (a,b,c,d,e)

Fig. 1.4

(3) $MTP(a, b, c, d, e, f)$ is the graph $(\{a, b, c, d, e, f\}, \{(a, b), (b, c), (c, d), (e, a), (e, b), (e, c), (e, d), (f, b), (f, c), (f, e)\})$. (Fig. 1.5).

(4) $C_n^{i_1, i_2, \cdots, i_k}(p_1, p_2, \cdots, p_n)$ is the graph on which $\{p_1, p_2, \cdots, p_{n-1}\}$ induces $P_{n-1}(p_1, p_2, \cdots, p_{n-1})$ and $N(p_n) = \{p_1, p_{n-1}\} \cup \{p_{i_1}, p_{i_2}, \cdots, p_{i_k}\}$ (See Fig. 1.5 for some



$C_n (v_1,v_2,v_3, \ldots ,v_n),\ n \geq 5$

$C_n^2 (v_1,v_2,v_3, \ldots ,v_n),\ n \geq 5$

$C_n^3 (v_1,v_2,v_3, \ldots ,v_n),\ n \geq 5$

$C_n^{2,3} (v_1,v_2,v_3, \ldots ,v_n),\ n \geq 5$

MTP (a,b,c,d,e,f)

Members of the family F_1 of graphs

Fig. 1.5

important examples).

In the illustrations of the graphs, a wavy line always indicates that an edge may or may not be present. Because they occur frequently, there are some special cases of the graphs in Fig. 1.5 which are singled out:

(5) $house\,(a,b,c,d,e) = C_5^3(a,b,c,d,e) = C_5^3(b,a,e,d,c)$ (Fig. 1.6).

(6) $tepee\,(a,b,c,d,e) = C_5^{2,3}(a,b,c,d,e)$ (Fig. 1.6).

(7) $step\text{-}up\,(a,b,c,d,e,f) = C_6^3(a,b,c,d,e,f)$ (Fig. 1.6).

In some instances it will be convenient to use a shorthand to refer to some of the above graphs. If one of the above graph names appears without arguments, then it will simply refer to a graph isomorphic to one of the same name *with* arguments. For example, a $C_n^{i_1, i_2, \cdots, i_k}$ is any graph isomorphic to $C_n^{i_1, i_2, \cdots, i_k}(1, 2, 3, \cdots, n)$. An *MTP* is any graph isomorphic to *MTP* (1,2,3,4,5,6), etc.

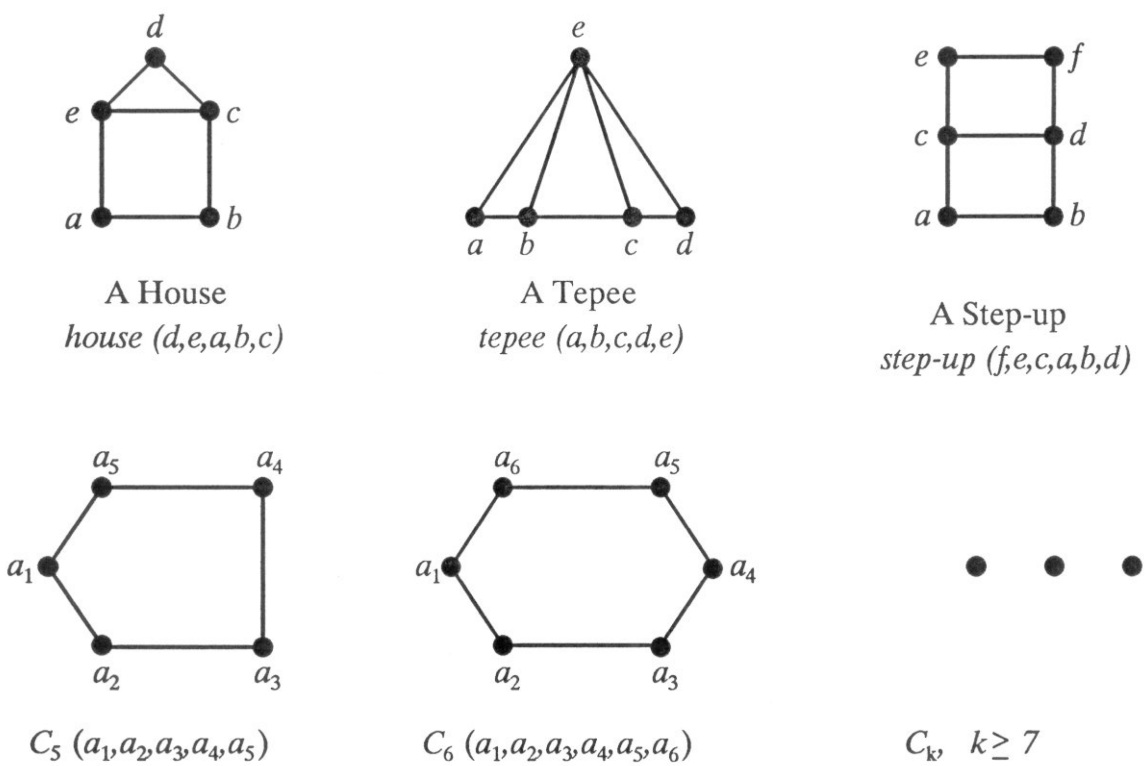


Fig. 1.6

Graphs in the set F.

It will be useful to designate two particular families of graphs so they may be referred to by name in the rest of this dissertation. The family of graphs, F, is the set of graphs made up of a *house*, *tepee*, *step−up*, and C_n for $n \geq 5$. These graphs are depicted in Fig. 1.6 The family of graphs, F_1, is the set of graphs made up of an *MTP* and C_n, C_n^2, C_n^3, and $C_n^{2,3}$ for $n \geq 5$. These graphs are depicted in Fig. 1.5.

The sequence $(V_1, \cdots, V_k = V)$ $(k \geq 1)$ is called a *constructing sequence* if G/V_i is prime for each $i \in \{1,2, \cdots ,k\}$, and $V_i \subseteq V_j$ for each $i<j \leq k$. A constructing sequence is called *basic* if the first set in the sequence contains no proper subset which induces a prime graph. A constructing sequence $(V_1, \cdots, V_k = V)$ is *maximal* if it is basic and for each $i \in \{1, 2, \cdots, k-1\}$, V_i is a proper subset of V_{i+1}, and if whenever W is a proper subset of V_{i+1} and V_i is a proper subset of W then G/W is not prime (in the appropriate sense). "Constructing" will be preceded by a number to indicate the type of primeness being considered (i.e. a 3*−constucting sequence* or 4*−constructing sequence*).

Let $W \subseteq V$. Throughout the remainder of this paper we will have occasion to refer to the following sets:

(1) For every $w \in W$, $M_W(w) = \{v \in V-W : N_W(v) - \{w\} = N_W(w)\}$

$$M_W = \bigcup_{w \in W} M_W(w)$$

(2) $A_W^0 = \{v \in V-W : N_W(v) = \varnothing\}$

(3) For every $w \in W$, $A_W^1(w) = \{v \in V-W : N_W(v) = \{w\}\}$

$$A_W^1 = \bigcup_{w \in W} A_W^1(w)$$

(4) $A_W^\infty = \{v \in V-W : N_W(v) = W\}$

(5) $A_W^{1+} = V - W - M_W - A_W^0 - A_W^\infty$

(6) $A_W^{2+} = V - W - M_W - A_W^0 - A_W^1$

In English, this means the following: $M_W(w)$ is simply the set of vertices in $V-W$ that are

W-similar to w. A_W^{∞} is the set of vertices in $V-W$ that are adjacent to every vertex in W. A_W^0 is the set of vertices in $V-W$ adjacent to no vertex in W. A_W^1 is the set of vertices in $V-W$ with exactly one neighbor in W. A_W^{1+} is the set of vertices in $V-W$ that are adjacent to at least one but not all the vertices in W but are *W-similar* to no vertex in W. A_W^{2+} is the set of vertices in $V-W$ that have at least two neighbors in W but are *W-similar* to no vertex in W.

If $G = (V, E)$ is a graph, then include a new vertex (call it w) which is adjacent to each vertex in G, and call this modified graph G' (formally $G' = (V \cup \{w\}, E \cup \{w\} \times V)$). The two notions of primeness mentioned above are related in the following way: G is 3-prime iff G' is 4-prime. The simplest example of a graph which is 3-prime but not 4-prime is a P_4 (Fig. 1.4). An example of a graph which is 4-prime but not 3-prime is a *tepee* (Fig. 1.6).

These notions have appeared in some previous papers under different names. The *4–decomposition* was originally developed by Cunningham [Cu] and was called a *split decomposition*; another name for it is *X–join* [Hs]. *3–decomposition* is also known as *substitution decomposition* [Mo], [Mr]; another name for it is *J–join* [Sp].

1.4. New Results

Cunningham [Cu] had developed an algorithm for *4–decomposition*, and this work (Sec. 2.3) improves the time analysis to $O(|V| \times |E|)$ and also offers a second (new) algorithm (Sec. 2.4 and 2.7) again achieving $O(|V| \times |E|)$ time. This second algorithm is used in proving additional theorems (3, 4, and 5).

The first two structural results (Theorems 1 and 2) in Chapter 2 (Sec. 2.2) are not new, but the corresponding algorithmic results are. These structural results are strengthened in a new way in Chapter 2.6 (theorems 3 and 4). Most structural results take the form (as do the first two in Chapter 2) that there is some subgraph contained within a given graph. The strengthened structural results are of the form that *each* vertex is contained within a particular subgraph of the given graph. In fact, the results may be strengthened even more to say that each pair of vertices (or each subset of vertices) must be contained within a particular subgraph of the given graph. The recognition algorithm for circle graphs (Chapter 3, Theorem 6) is new and presents a natural correspondence between circle graphs and 4–*prime* graphs.

2. Graph Decompositions

2.1. Overview of Chapter 2

This chapter will establish certain properties of graphs which are tied to decompositions of graphs. Most of the properties concern induced subgraphs of the graphs in question. Thus, we first show that a graph with no similar pairs must contain an induced P_4 (Theorem 1). Then we show (Theorem 2) that a graph with no similar pairs and no articulation points must contain one of the graphs from Fig. 1.6 as an induced subgraph. Following this are two decomposition theorems for graphs. The first is merely an improved analysis of Cunningham's original algorithm; the second (Central Lemma) is the basis for the following proofs in this dissertation including recognition of circle graphs. Immediate use of this lemma (1) is made in refining the results of the first two theorems.

2.2. Initial Characterizations

THEOREM 1: Let $G = (V, E)$, $|V| \geq 2$, be a graph having no similar pairs. Then some induced subgraph of G is a P_4. Furthermore, such a subgraph can be found in $O(|E|)$ time. (Note that the absence of similar pairs implies $|E| = \Omega(|V|)$.)

PROOF: For all $k \geq 0$, let H_k denote the graph with vertices $\{a_i : 0 \leq i \leq 2k+1\}$, and edges $\{(a_i, a_j) : i \neq j$ and j is even and (i is even or $i > j$)$\}$ (see Fig. 2.1). Our algorithm iteratively finds an induced subgraph of G isomorphic to H_k for $k = 0, 1, \cdots$ until eventually finding an induced P_4.

More precisely, we initially chose some edge $(w_0, w_1) \in E$ (E cannot be empty since otherwise each vertex of G would be isolated and hence, since $|V| \geq 2$, G would have a similar pair). Thus $G/\{w_0, w_1\}$ is isomorphic to H_0.

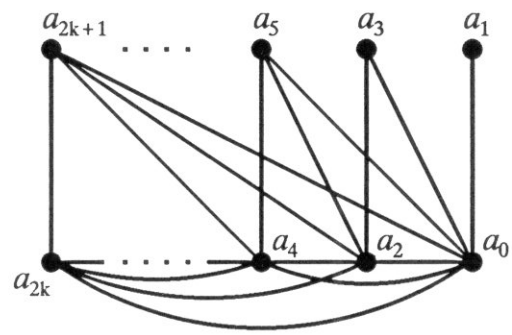


Fig. 2.1
The graph H_k

Now assume that we have found a set $W = \{w_j : 0 \le j \le 2k+1\} \subseteq V$ that induces a subgraph of G isomorphic to H_k, for some $k \ge 0$, where w_j corresponds to a_j under the isomorphism. Note that since w_{2k} and w_{2k+1} are W-similar there must exist some $z_0 \in V - W$ adjacent to exactly one of them (otherwise w_{2k} and w_{2k+1} would be V-similar). Assume that z_0 is adjacent to w_{2k} but not to w_{2k+1}; this is without loss of generality by the W-similarity of w_{2k} and w_{2k+1}. Now if z_0 is adjacent to w_{2j+1} for some $j, 0 \le j < k$, then $\{w_{2j+1}, z_0, w_{2k}, w_{2k+1}\}$ induces a P_4 (and so we halt and output it). Otherwise if z_0 is not adjacent to w_{2j} for some $j, 0 \le j < k$, then $\{z_0, w_{2k}, w_{2j}, w_{2j+1}\}$ induces a P_4. So assume that neither of these two cases holds.

At this point we know that z_0 and w_{2k+1} are $W \cup \{z_0\}$-similar. Therefore there must be some $z_1 \in V - (W \cup \{z_0\})$ adjacent to exactly one of $\{z_0, w_{2k+1}\}$, for otherwise z_0 and w_{2k+1} would be similar. Since z_0 and w_{2k+1} are $W \cup \{z_0\}$-similar, it is without loss of generality to assume that z_1 is adjacent to z_0 but not to w_{2k+1}. Now if z_1 is not adjacent to w_{2j} for some $j, 0 \le j \le k$, then $\{z_1, z_0, w_{2j}, w_{2k+1}\}$ induces a P_4. Otherwise if z_1 is adjacent to w_{2j+1} for some $j, 0 \le j < k$, then $\{w_{2j+1}, z_1, w_{2k}, w_{2k+1}\}$ induces a P_4.

Thus if we have not yet found an induced P_4 in G then the set $W \cup \{z_0, z_1\}$ induces a subgraph isomorphic to H_{k+1}.

By induction, the algorithm must find an induced P_4 in G, since otherwise it finds an induced subgraph in G isomorphic to H_k for each $k \geq 0$, contradicting the finiteness of G.

We now describe how to implement the algorithm in the specified time bound. At the kth iteration we must find a vertex z_0 adjacent to w_{2k}, but not to w_{2k+1}, or vice versa. To do this, we can traverse the adjacency list for w_{2k}, placing a marker in the slots corresponding to its elements in an array representing all the members of V. Then we traverse the adjacency list for w_{2k+1}; if we ever find an unmarked vertex in it, we halt and return that as z_0 (since it is adjacent to w_{2k+1} but not to w_{2k}). Similarly, we can search for a vertex adjacent to w_{2k} but not to w_{2k+1}). We check whether z_0 is adjacent to w_{2j+1} for some j, $0 \leq j < k$, by scanning the adjacency list for z_0. We check whether z_0 is not adjacent to w_{2j} for some $0 \leq j < k$ by scanning the adjacency list for z_0 and marking off the even-indexed elements of W as they are encountered --- thus the work is $O(\deg(z_0))$, unless this is the last iteration, in which case it is $O(|V|)$. Finding z_1 and checking its adjacencies is done analogously. Therefore the total time required is $O(|E|)$.

QED Theorem 1

Actually, Theorem 1 is not new (although this proof is), because the existence of the induced P_4 follows from [CL]; one can then apply the algorithm of [CP] that finds an induced P_4 (whenever it exists) in a graph (V, E) in $O(|V| + |E|)$ time.

THEOREM 2: Let $G = (V, E)$, $|V| \geq 2$, be a graph having neither similar pairs nor articulation points. Then some induced subgraph of G is a member of **F** (the set of graphs in Fig. 1.6). Furthermore, such a subgraph can be found in $O(|E|)$ time.

PROOF: Our algorithm to find such an induced subgraph first finds a P_4 (Step 1). Then in each of Steps 2 through 9, we test for some condition and if it is satisfied then we can find a member of **F** and halt; otherwise we continue, and utilize the falsity of the condition in subsequent steps. Finally (Step 10), if none of the conditions has been met, then we can find a set $U \subseteq V$ such that

(1) there is a particular vertex w that is adjacent to each member of U, (2) $|U| \geq 2$, and (3) U contains no U-similar pairs. Therefore we can simply apply Theorem 1 to the subgraph G/U to obtain another P_4; this P_4 together with w induces a *tepee*.

More precisely, the algorithm is:

Step 1: Find vertices $a, b, c, d \in V$ that induce a $P_4(a, b, c, d)$ by means of the algorithm in the proof of Theorem 1.

Step 2: If there exists $v \in V$ adjacent to both a and d then output the subgraph induced by $\{a, b, c, d, v\}$ (which must be a *house*, *tepee* or C_5) and halt. In the remainder, we therefore assume that $\text{dist}(a, d) = 3$, where *dist* denotes the length of the shortest path in the graph between the two vertices.

Step 3:

$B \leftarrow \{b' \in V : (a, b') \in E$ and there exists $c' \in V$ such that $(b', c'), (c', d) \in E\}$;

$C \leftarrow \{c' \in V : (d, c') \in E$ and there exists $b' \in V$ such that $(a, b'), (b', c') \in E\}$.

Thus, B is the set of vertices adjacent to a in the set of induced paths of length 3 from a to d. Similarly, C is the set of vertices adjacent to d in the set of induced paths of length 3 from d to a. If there exist vertices $b_1 \in B, c_1 \in C$ such that $(b_1, c_1) \notin E$ then we can find an induced member of **F** as follows. Since $b_1 \in B, c_1 \in C$, there exist vertices b_2, c_2 such that (a, b_2), $(b_2, c_1) \in E$, and (b_1, c_2), $(c_2, d) \in E$ (Fig. 2.2.1). Note that Step 2 ensures that (a, c_2), $(b_2, d) \notin E$. There are eight cases depending on which subset of $\{(b_1, b_2), (b_2, c_2), (c_1, c_2)\}$ is contained in E. The reader may verify that there is an induced member of **F** in each case.

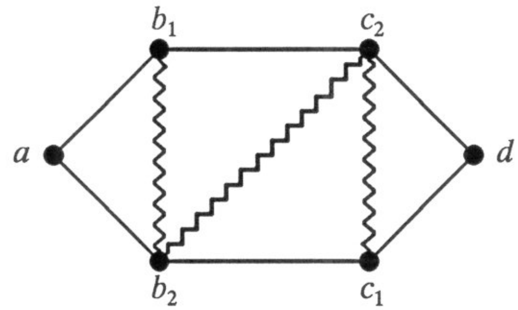


Fig. 2.2.1

Thus

FACT 1: for all $b_1 \in B$, $c_1 \in C$, $(b_1, c_1) \in E$ (see Fig. 2.2.2).

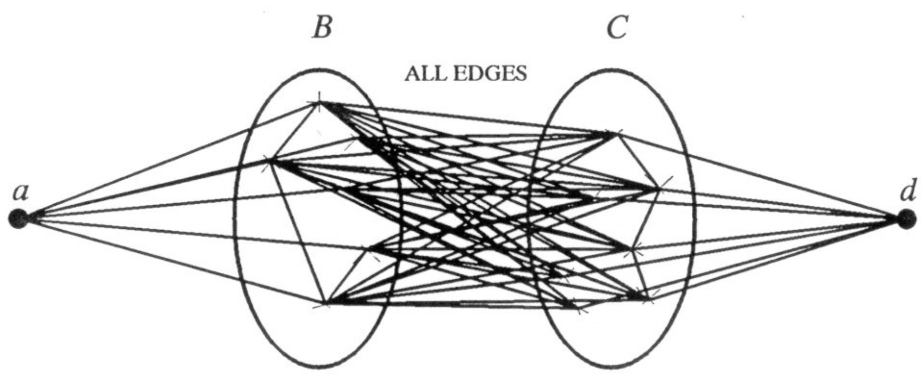


Fig 2.2.2

Step 4: If $|B| = 1$ then let $P_{r+1}(a = p_0, p_1, p_2, \cdots, p_r = c)$ be the shortest path from a to c not containing b (such a path must exist, otherwise b would be an articulation point). We know that $r \geq 3$, since otherwise B would have at least two elements, namely b and p_1. Consider the sub-graph induced by $\{b, a, p_1, p_2, \cdots, p_{r-1}, c\}$; it must be of the form shown in Fig. 2.2.3. Let k be the smallest integer greater than 2 such that p_k is adjacent to b.

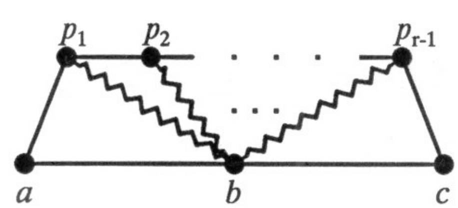


Fig. 2.2.3

We consider four cases:

CASE 1: $(p_1, b), (p_2, b) \notin E$.

Then $G/\{a, p_1, p_2, \cdots, p_k, b\}$ is a C_{k+2} where $k+2 \geq 5$.

CASE 2: $(p_1, b) \in E, (p_2, b) \notin E$.

If $k = 3$ then $\{a, p_1, p_2, p_3, b\}$ induces a *house*. If $k \geq 4$ then $G/\{p_1, p_2, \cdots, p_k, b\}$ is a C_{k+1} where $k+1 \geq 5$.

CASE 3: $(p_1, b) \notin E, (p_2, b) \in E$.

If $k = 3$ then $\{a, p_1, p_2, p_3, b\}$ induces a *house*. If $k = 4$ then $\{a, p_1, p_2, p_3, p_4, b\}$ induces a *step−up*. If $k \geq 5$ then $\{p_2, p_3, \cdots, p_k, b\}$ induces a C_k

CASE 4: $(p_1, b), (p_2, b) \in E$.

If $k = 3$ then $\{a, p_1, p_2, p_3, b\}$ induces a *tepee*. If $k = 4$ then $\{p_1, p_2, p_3, p_4, b\}$ induces a *house*. If $k \geq 5$ then $\{p_2, p_3, \cdots, p_k, b\}$ induces a C_k.

Step 5:

$W_1 \leftarrow$ $\{v \in V - B : v$ is adjacent to at least one element of B and at least one element of $C \cup \{a, d\}\}$;

$W_2 \leftarrow$ $\{v \in V - B : v$ is adjacent to at least one element of B but no elements of $C \cup \{a, d\}\}$.

Thus $\{W_1, W_2\}$ is a partition of the vertices outside of B but adjacent to elements of B.

After Step 6, each element of W_1 will be adjacent to each element of B. After Step 7, each element of W_2 will be adjacent to at least two elements of B. Steps 8 and 9 will use these two facts to produce a set $U \subseteq B$ having at least two elements and no U-similar pairs, which then allows us to find a member of **F** in Step 10.

Step 6: If there exists $w \in W_1$ such that $N_B(w) \neq B$ then we find a member of **F** as follows.

First note that it cannot be that $N_C(w) \neq \varnothing$ and $(w,a) \in E$, otherwise we would have $w \in B$. Furthermore, it cannot be that $N_C(w) = \varnothing$ and $(w,a) \notin E$, otherwise we would have $(w,d) \in E$ and hence (since $N_B(w) \neq \varnothing$) we would have $w \in C$. Also note that $w \neq a$, since otherwise dist$(a, b) \leq 2$. Since w is adjacent to at least one, but not all elements of B, there exists $b_1, b_2 \in B$ such that w is adjacent to b_1 but not to b_2.

CASE 1: $(w,a) \notin E, c_1 \in N_C(w)$ (Fig. 2.2.4)

$\{b_1, a, b_2, c_1, w\}$ induces a *tepee* if $(b_1, b_2) \in E$, and a *house* otherwise.

CASE 2: $N_C(w) = \varnothing, (w,a) \in E$ (Fig. 2.2.5).

$\{b_1, c, b_2, a, w\}$ induces a *tepee* if $(b_1, b_2) \in E$, and a *house* otherwise.

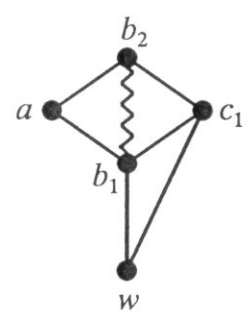


Fig. 2.2.4

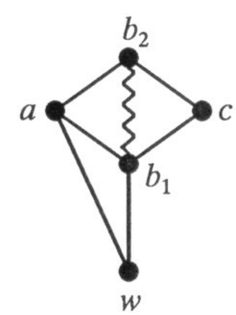


Fig. 2.2.5

Step 7:

$$Q \leftarrow \{w \in W_2 : |N_B(w)| = 1\}.$$

If Q is non-empty then we find a member of **F** as follows. For each $q \in Q$ define b_q as the sole element of $N_B(q)$, and define path(q) as a shortest path not containing b_q from q to some $b' \in B - \{b_q\}$ (such a path must exist since $|B| \geq 2$ by Step 4 and b_q is not an articulation point. Find some $q_0 \in Q$ such that path(q_0) has minimum length; let $P_{k+2}(q_0, p_1, p_2, \cdots, p_k, b')$ be path(q_0). Since $p_i \notin B$ for $1 \leq i \leq k$, p_i is adjacent to at most one of a or c.

We must have $k \geq 1$, since $|N_B(q_0)| = 1$. Also note that $p_1 \notin \{a, c\}$ since $q_0 \in W_2$.

CASE 1: $k = 1$.

If $(a, p_1) \in E$, then (as shown above) p_1 is not adjacent to c. q_0 is not adjacent to c since $q_0 \in W_2$. Thus (Fig. 2.2.6) $\{b_{q_0}, q_0, p_1, b', c\}$ induces either a *tepee* (if b_{q_0} is adjacent to both p_1 and b'), a *house* (if it is adjacent to exactly one of them), or a C_5 (if it is adjacent to neither). On the other hand, if $(a, p_1) \notin E$ (Fig. 2.2.7), then we have that $\{a, b', b_{q_0}, q_0, p_1\}$ induces either a *tepee*, *house* or C_5 in each of the four cases depending on which elements of $\{b', p_1\}$ are adjacent to b_{q_0}.

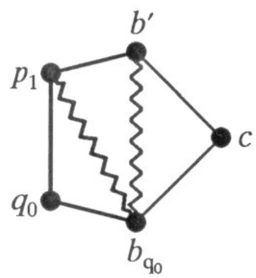


Fig. 2.2.6

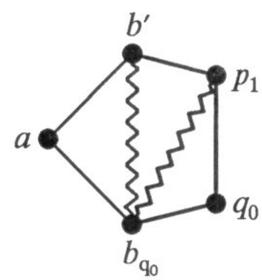


Fig. 2.2.7

The following two facts are used in cases 2 and 3:

FACT 2: for all i, $1 \leq i \leq k-2$, p_i is adjacent to neither a nor c, since otherwise there would be a path from q_0 to b' shorter than path(q_0) (b' is adjacent to c by Fact 1).

FACT 3: for all i, $1 \leq i \leq k-1$, p_i is not adjacent to b_{q_0}. To see this, assume that it is. If $p_i \in W_1$ then $(p_i, b) \in E$, contradicting the choice of path(q_0). If $p_i \in W_2 - Q$ then the choice of path(q_0) is again violated. Otherwise the choice of q_0 is violated.

CASE 2: $k = 2$.

CASE 2.1: $(b_{q_0}, b') \in E$ (Fig. 2.2.8).

Then $\{b_{q_0}, q_0, p_1, p_2, b'\}$ induces a *house* if $(b_{q_0}, p_2) \in E$ and a C_5 otherwise.

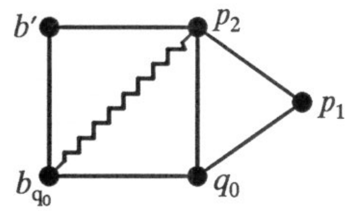


Fig. 2.2.8

CASE 2.2: $(b_{q_0}, b') \notin E$.

CASE 2.2.1: $(p_1, a) \in E$.

Then $(p_1, c) \notin E$ since $p_1 \notin B$ (Fig. 2.2.9) so $\{b_{q_0}, q_0, p_1, a, b', c\}$ induces a *step−up*.

CASE 2.2.2: $(p_1, a) \notin E$ (Fig. 2.2.10).

Then if $(p_2, a) \in E$ then $\{b_{q_0}, q_0, p_1, p_2, a\}$ induces a *house* or a C_5 (depending on whether $(p_2, b_{q_0}) \in E$). On the other hand, if $(p_2, a) \notin E$ then $\{b_{q_0}, q_0, p_1, p_2, b', a\}$ induces a *step−up* or a C_6 (depending on whether $(p_2, b_{q_0}) \in E$).

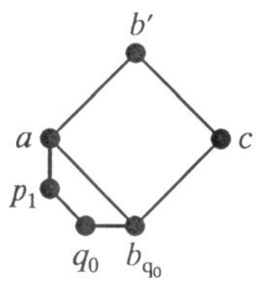


Fig. 2.2.9

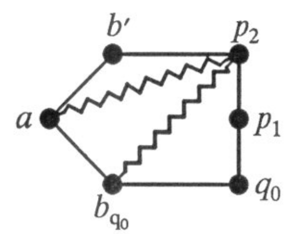


Fig. 2.2.10

CASE 3: $k \geq 3$.

CASE 3.1: $(p_{k-1}, c) \in E$ (Fig. 2.2.11).

Then $\{b_{q_0}, q_0, p_1, p_2, \cdots, p_{k-1}, c\}$ is a C_{k+2}, where $k+2 \geq 5$.

CASE 3.2: $(p_{k-1}, c) \notin E$.

CASE 3.2.1: $(p_k, b_{q_0}) \in E$.

Then $G/\{b_{q_0}, q_0, p_1, p_2, \cdots, p_k\}$ is a C_{k+2} where $k+2 \geq 5$.

CASE 3.2.2: $(p_k, b_{q_0}) \notin E$ (Fig. 2.2.12).

If $(p_k, c) \in E$ then $G/\{b_{q_0}, q_0, p_1, p_2, \cdots, p_k, c\}$ is a C_{k+3} where $k+3 \geq 6$. Otherwise either $G/\ \{b_{q_0}, q_0, p_1, p_2, \cdots, p_k, b\}$ is a C_{k+3} where $k+3 \geq 6$ (if $(b', b_{q_0}) \in E$) or $G/\ \{b_{q_0}, q_0, p_1, p_2, \cdots, p_k, b, c\}$ is a C_{k+4} where $k+4 \geq 7$ (if $(b', b_{q_0}) \notin E$).

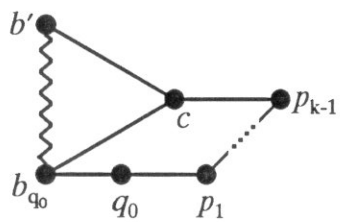


Fig. 2.2.11

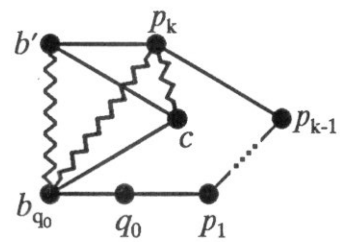


Fig. 2.2.12

Thus, assume in the remainder $|N_B(w)| \geq 2$ for all $w \in W_2$.

Step 8:

$w \leftarrow$ an element of W_2 having the fewest neighbors in B, i.e. $|N_B(w)| \leq |N_B(w')|$ for all $w' \in W_2$ (We know $W_2 \neq \emptyset$ since $a \in W_2$);

$U \leftarrow N_B(w)$.

If there exists $v \in B - U$ adjacent to some $b_1 \in U$ but not adjacent to some $b_2 \in U$ then (Fig. 2.2.13) $\{b_1, w, b_2, c, v\}$ induces a *tepee* if $(b_1, b_2) \in E$ and a *house* otherwise.

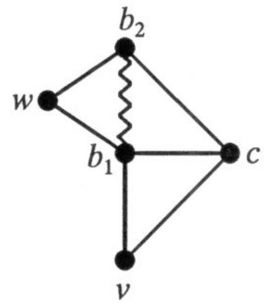


Fig. 2.2.13

Thus, suppose each vertex in B outside of U is adjacent to either none or all of the vertices in U.

Step 9: If there exists $w' \in W_2$ adjacent to some $b_1 \in U$ but not adjacent to some $b_2 \in U$, then we can find a member of **F** as follows. Let b_3 be an element of $B - U$ that is adjacent to w' (such a vertex must exist, since otherwise $|N_B(w)| = |U| > |U - \{b_2\}| \geq |N_B(w')|$ contradicting the choice of w).

CASE 1: $(w, w') \in E$ (Fig. 2.2.14).

Then $\{b_1, a, b_2, w', w\}$ induces a *tepee* if $(b_1, b_2) \in E$ and a *house* otherwise.

CASE 2: $(w, w') \notin E$ (Fig. 2.2.15).

CASE 2.1: $(b_2, b_3) \in E$.

Then $\{b_1, w, b_2, b_3, w'\}$ induces a *tepee* (if b_1 is adjacent to both b_2 and b_3), a C_5 (if b_1 is adjacent to neither b_2 nor b_3), or a *house* otherwise.

CASE 2.2: $(b_2, b_3) \notin E$.

If b_1 is adjacent to both b_2 and b_3 then $\{b_1, w, b_2, a, b_3\}$ induces a *tepee*. If b_1 is adjacent to b_2 but not b_3 then $\{b_1, w', b_3, a, b_2\}$ induces a *house*. If b_1 is adjacent to b_3 but not b_2 then $\{b_1, w, b_2, a, b_3\}$ induces a *house*. Finally, if b_1 is adjacent to neither b_2 nor b_3 then $\{b_1, w, b_2, a, b_3, w'\}$ induces a *step-up*.

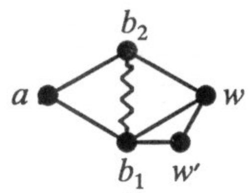


Fig. 2.2.14

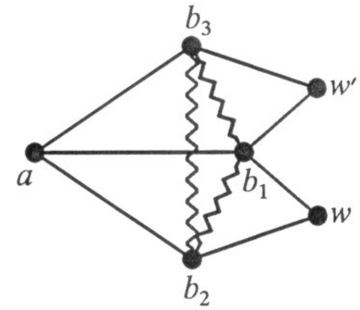


Fig. 2.2.15

FACT 4: U has no U-similar pairs.

PROOF: We claim that each vertex $v \in V - U$ is adjacent to all or to none of the vertices in U. Step 6 ensures this for each $v \in W_1$ (since $U \subseteq B$). Step 8 ensures this for each $v \in B - U$. Step 9 ensures this for each $v \in W_2$. Therefore, since there are no V-similar pairs, if $u_1, u_2 \in U$, they are not a U-similar pair. **QED**

Step 10: Find vertices $u_1, u_2, u_3, u_4 \in U$ that induce a P_4, by again using the algorithm of the proof of Theorem 1 (Facts 4 and 5 allow us to do this). Thus $\{u_1, u_2, u_3, u_4, w\}$ induces a *tepee*.

It is a simple matter to implement Steps 1-6 and 8-10 in $O(|E|)$ time. For Step 7 it is only slightly more complicated: recall that there we must find a shortest path $(q_0, p_1, p_2, \cdots, p_k, b')$ subject to $q_0 \in Q$, $p_i \neq b_{q_0}$ for all $1 \leq i \leq k$, and $b' \in B - \{b_{q_0}\}$. To do this, we first find a shortest path P_1 of this form but subject also to $p_k \notin Q$. Secondly, we find a shortest path $(q, p_1, p_2, \cdots, p_k)$ subject to $q, p_k \in Q$, $b_q \neq b_{p_k}$; and then let P_2 be the path $(q, p_1, p_2, \cdots, p_k, b_{p_k})$. Both P_1 and P_2 can be found in $O(|E|)$ by means of breadth-first search. We then choose the shorter of P_1 and P_2 as our desired path.

QED Theorem 2

The existence result in Theorem 2 follows from the independently discovered results of [BM]. In particular, a graph having neither similar pairs nor articulation points cannot be "distance-hereditary" in their sense, and hence must contain an induced member of **F**. Their paper does not seem to provide the appropriate algorithmic result.

2.3. 4-Factoring a Graph

We now present an O($|V| \times |E|$)-time algorithm to find a 4-decomposition of an undirected graph G, thus improving on the O($|V|^3$)-time algorithm of [Cu]. As is shown in [Cu], this problem can be solved by making O($|V|$) calls to a subroutine that solves the following problem:

INPUT: a graph $G = (V, E)$ and an edge $(x, y) \in E$.

OUTPUT: a partition $\{V_0, V_1, V_2, V_3\}$ of V yielding a 4-factoring, such that $x \in V_1$ and $y \in V_2$, if such a partition exists; otherwise output "no".

An O($|V|^2$)-time algorithm to solve this problem is given in [Cu]; our algorithm, which runs in O($|E|$) time, is shown in Fig. 2.3. We maintain a partition $\{S, T\}$ of V, such that $x \in S$ and $y \in T$. We try to construct these sets so that there is a partition $\{V_0, V_1, V_2, V_3\}$ of V yielding a 4-factoring, where $S = V_0 \cup V_1$ and $T = V_2 \cup V_3$; if such a 4-factoring exists then we refer to the set $\{S, T\}$ as a *split*. Initially S contains only x and one other (arbitrarily chosen) vertex w, and T contains all other vertices. Define a *violation* as a pair $\{s, t\}$ such that $s \in S$, and $t \in T$ and one of the following four cases is true (see Fig. 2.4)

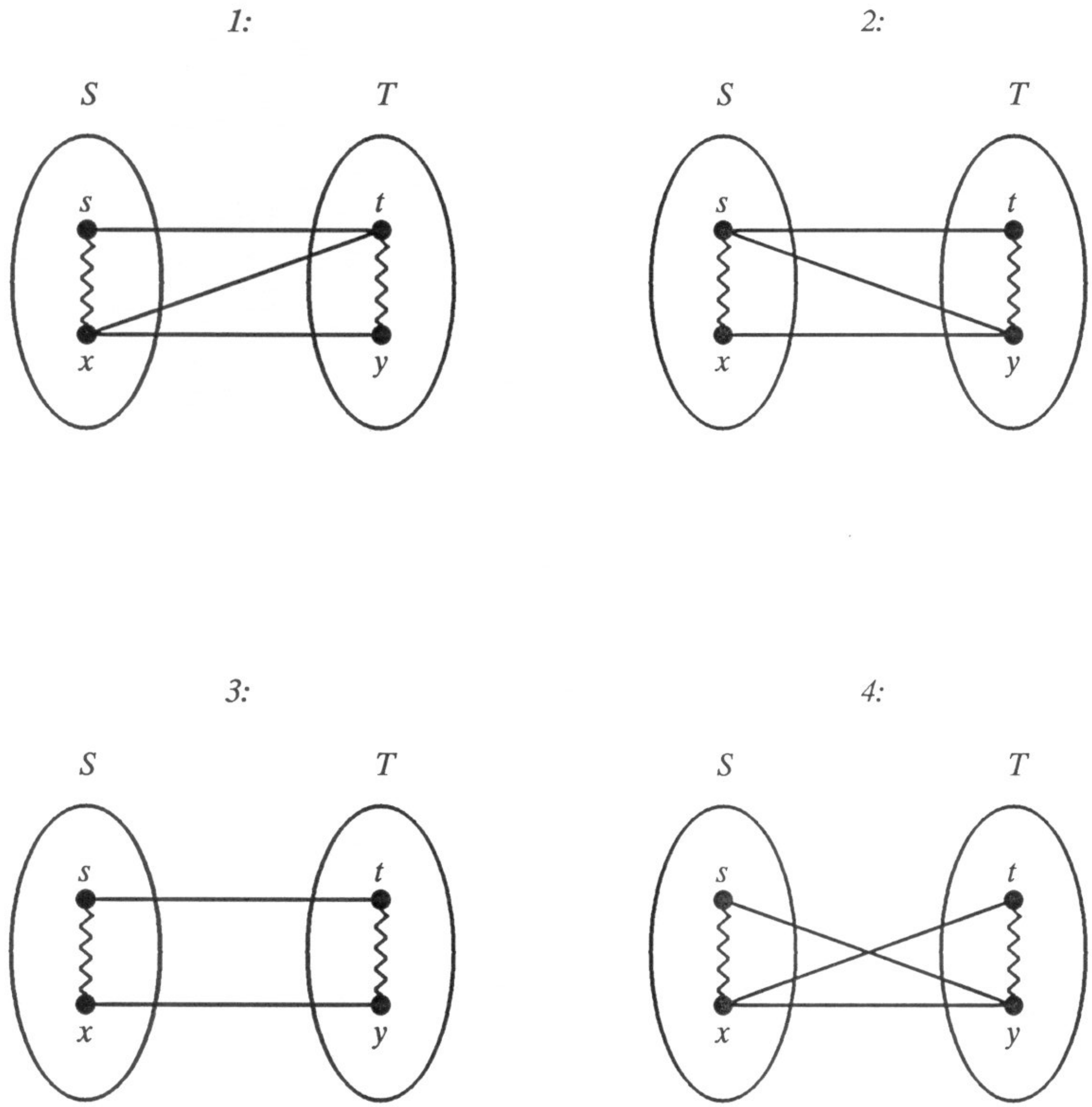
1. $(s, t) \in E$, $(s, y) \notin E$, $(t, x) \in E$.
2. $(s, t) \in E$, $(s, y) \in E$, $(t, x) \notin E$.
3. $(s, t) \in E$, $(s, y) \notin E$, $(t, x) \notin E$.
4. $(s, t) \notin E$, $(s, y) \in E$, $(t, x) \in E$.

It is easily verified that if there is a violation then $\{S, T\}$ is not a split. The algorithm iteratively looks for violations, and whenever it finds one, eliminates it by moving the member of T involved in the violation into S. We use a set U to hold all vertices s that have been moved into S, but that we have not yet examined to see whether there is a $t \in T$ with which s forms a violation.

```
(1) w ← some element of V − {x,y} ;
(2) S ← {x,w} ;
(3) T ← V − S ;
(4) U ← {w} ;
(5) WHILE U ≠ ∅ DO
(6)     BEGIN
(7)       s ← some member of U ; U ← U − {s} ;
(8)       [[ Look for violations of the form {s,t} where t ∈ T ]]
(9)         IF (s,y) ∈ E
(10)            THEN FOR each t ∈ N_T(s) ∪ N_T(x) DO
(11)                   IF {s,t} is a violation THEN
(12)                      BEGIN
(13)                         T ← T − {t} ;
(14)                         S ← S ∪ {t} ;
(15)                         U ← U ∪ {t}
(16)                      END
(17)            ELSE FOR each t ∈ N_T(s) DO
(18)                   BEGIN
(19)                      [[ {s,t} must be a violation ]]
(20)                        T ← T − {t} ;
(21)                        S ← S ∪ {t} ;
(22)                        U ← U ∪ {t}
(23)                   END;
(24)     END;
(25) [[ Now there are no violations ]]
(26) IF |T| > 1
(27)   THEN BEGIN
(28)          V_1 ← {s ∈ S : (s,y) ∈ E};
(29)          V_0 ← S − V_1;
(30)          V_2 ← {t ∈ T : (t,x) ∈ E};
(31)          V_3 ← T − V_2;
(32)          output {V_0, V_1, V_2, V_3}
(33)        END
(34)   ELSE
(35)     Interchange x and y and repeat the WHILE loop with S = {y,w} and T = V − S .
(36)     Again, if |T|>1 then output a partition as in steps (28)-(32); otherwise output "no".
```

Fig. 2.3

The algorithm (A) to check for a 4-factoring splitting a particular edge.



The four types of violations.

A straight line indicates an edge, absence of a line indicates no edge, and a wavy line indicates that an edge may or may not be present.

Fig. 2.4

PROOF OF CORRECTNESS:

If there is a split $\{S, T\}$ with $x \in S, y \in T$ then either $w \in S$ or $w \in T$. Hence it suffices to check that the algorithm correctly determines whether there is a split $\{S, T\}$ with $\{x, w\} \subseteq S$, $y \in T$. We show first that after the termination of the WHILE loop there is no violation. Assume for a contradiction that there were such a violation $\{s, t\}$. Since every element of S other than x was removed from U at some point, we can consider the point at which s was removed from U. If $(s, y) \in E$ then (since $\{s, t\}$ is a violation) we must have that $t \in N_T(s) \cup N_T(x)$. But this implies that the violation $\{s, t\}$ would have been detected in step (10), and therefore t would have been included in S, a contradiction. On the other hand, if $(s, y) \notin E$ then the violation $\{s, t\}$ would have been detected in step (17) and therefore t would have been included in S, again a contradiction.

Now if $|T| > 1$ then it is easy to verify that the partition $\{V_0, V_1, V_2, V_3\}$ yields a 4-factoring. On the other hand, if there is a split $\{S', T'\}$ with $x \in S', y \in T'$ then no element of S' could constitute a violation with an element of T.

ANALYSIS OF TIME COMPLEXITY:

For each vertex $v \in V$, we maintain a doubly-linked list of its neighbors in T (i.e. the set $N_T(v)$). When an element t is removed from T, it must be deleted from each of the deg(t) such lists. To facilitate this, we also maintain, for each $t \in T$, a list of the nodes corresponding to it in these lists; thus there is one such node (in the list for $N_T(v)$) for each v adjacent to t.

Note that no vertex is removed from U more than once. Furthermore, we claim that whenever an element s is removed from U, the amount of time spent in checking for violations involving s (which is proportional to the number of executions of the FOR loops in steps (10) and (17)) is

$$O(\deg(s) + \#\text{ of violations found involving } s)$$

To see this, note that step (10) examines some $t \in T$ either when $t \in N_T(s)$ (which happens for at most deg(s) values of t), or when $t \in N_T(x) - N_T(s)$ (which implies that $\{s, t\}$ is a violation, of type 4). To enumerate the elements in the set $N_T(x) - N_T(s)$ efficiently, we first traverse the list $N_T(s)$, marking each element; we then traverse the list $N_T(x)$ reporting each unmarked element. These two traversals take O(deg(s) + # of violations found involving s) time. Finally, step (17) executes $|N_T(s)|$ ($\leq$ deg(s)) times.

Thus, the running time of the algorithm is at most

$$\sum_{s \in V - \{x,y\}} \mathrm{O}(\deg(s) + \#\text{ of violations found involving } s\ )$$

$$= \mathrm{O}(|V| + |E|) = \mathrm{O}(|E|),$$

since the number of violations found is at most $|V| - 2$ (since each such violation causes an element to be removed from T).

In summary, the running time of algorithm of [Cu] to find a 4-decomposition is dominated by O($|V|$) calls to this algorithm; thus its total time is O($|V| \times |E|$).

2.4. Central Lemma

Although the previous algorithm (A, page 26) allowed us to 4-decompose a graph into its 4-prime components, we have found the following Lemma and its derivatives much more helpful in proving additional properties about prime graphs, and in solving the recognition problem on circle and permutation graphs. There are several different forms to this lemma, and we shall present the most general one (Lemma 1) first, and then show how to get some more specific versions. What each variation of Lemma 1 states is that starting from a prime, induced subgraph G/W, there is a procedure whereby W may be augmented by a few vertices in such a manner as to produce a larger induced prime subgraph of G (which also contains G/W as an induced subgraph), unless G itself is not prime. In this case, the procedure will naturally fail at some point.

Each lemma (1,2, and 3 along with Cor. 1 and 2) can be thought of as a set of rules whereby, starting from an initial vertex set V_0 such that $G/V_0 \in F$, a basic constructing sequence (recall the def. in Sec 1), $(V_0, V_1, \cdots, V_k)$, is formed where each graph G/V_i is formed from G/V_{i-1} by application of one of the rules given in the lemma. In this case, we say that $(V_0, V_1, \cdots, V_k)$ is a constructing sequence subject to the lemma.

All of the lemmas rely heavily on the use of the definitions given in the Introduction. To facilitate understanding the lemmas, it is important to establish a few facts.

FACT 5: If $G = (V, E)$ is 4-prime then (1) G has no articulation points, and (2) G has no similar pairs.

PROOF: (1) If G has an articulation point v, then let $\{A, B\}$ be a partition of $V - \{v\}$ sets such that $|B| \geq |A| \geq 1$ (hence $|B| \geq 2$) and the removal of v from G leaves G/A disconnected from G/B. Then the partition of V into

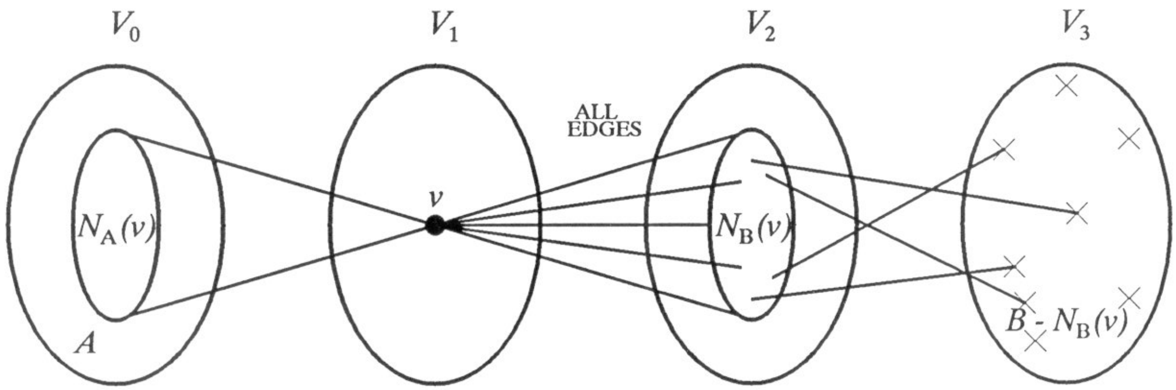
$$V_0 = A$$

$$V_1 = \{v\}$$

$$V_2 = N_B(v)$$

$$V_3 = B - N_B(v)$$

yields a 4-factoring of G (Fig. 2.5).



4-Factoring a graph with articulation point, v

Fig 2.5

(2) If $a, b \in V$ are a similar pair then the partition of V into

$$V_0 = V - (N_V(a) \cup \{a, b\})$$

$$V_1 = N_V(a) - \{b\}$$

$$V_2 = \{a, b\}$$

$$V_3 = \emptyset$$

yields a 4-factoring of G (Fig. 2.6). **QED**

FACT 6: If $G = (V, E)$ is a 3-prime graph having at least three vertices then G has no similar pairs.

PROOF: If $a, b \in V$ are a similar pair then the partition of V into

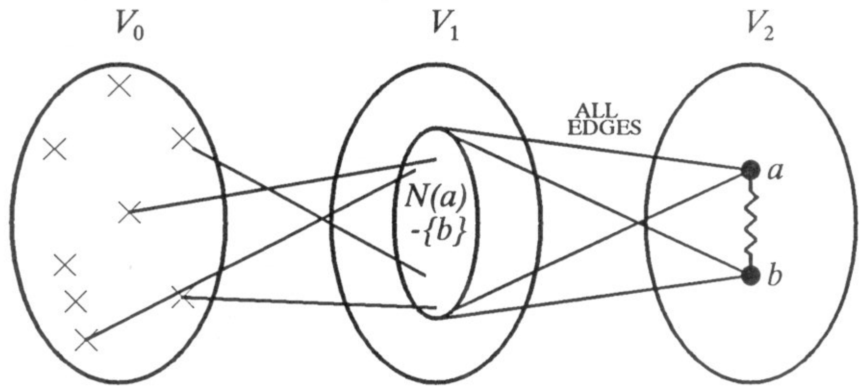
$$V_0 = V - (N_V(a) \cup \{a, b\})$$

$$V_1 = N_V(a) - \{b\}$$

$$V_2 = \{a, b\}$$

yields a 3-factoring of G (Fig. 2.6). **QED**

It is easy to verify that there are no 3-prime graphs of size 3, and there are are no 4-prime graphs of size 4. All 3-prime graphs contain an induced P_4 (by Theorem 1), and all 4-prime



Factoring a graph with a similar pair, *a* and *b*

Fig 2.6

graphs contain a member of **F** as an induced subgraph (by Theorem 2). From the fact that there are no similar pairs in either 3-prime or 4-prime graphs we have the following two facts:

Fact 7: If G/W is 3-prime, and $w_1, w_2 \in W$ such that $w_1 \neq w_2$ then $M_W(w_1) \cap M_W(w_2) = \emptyset$.

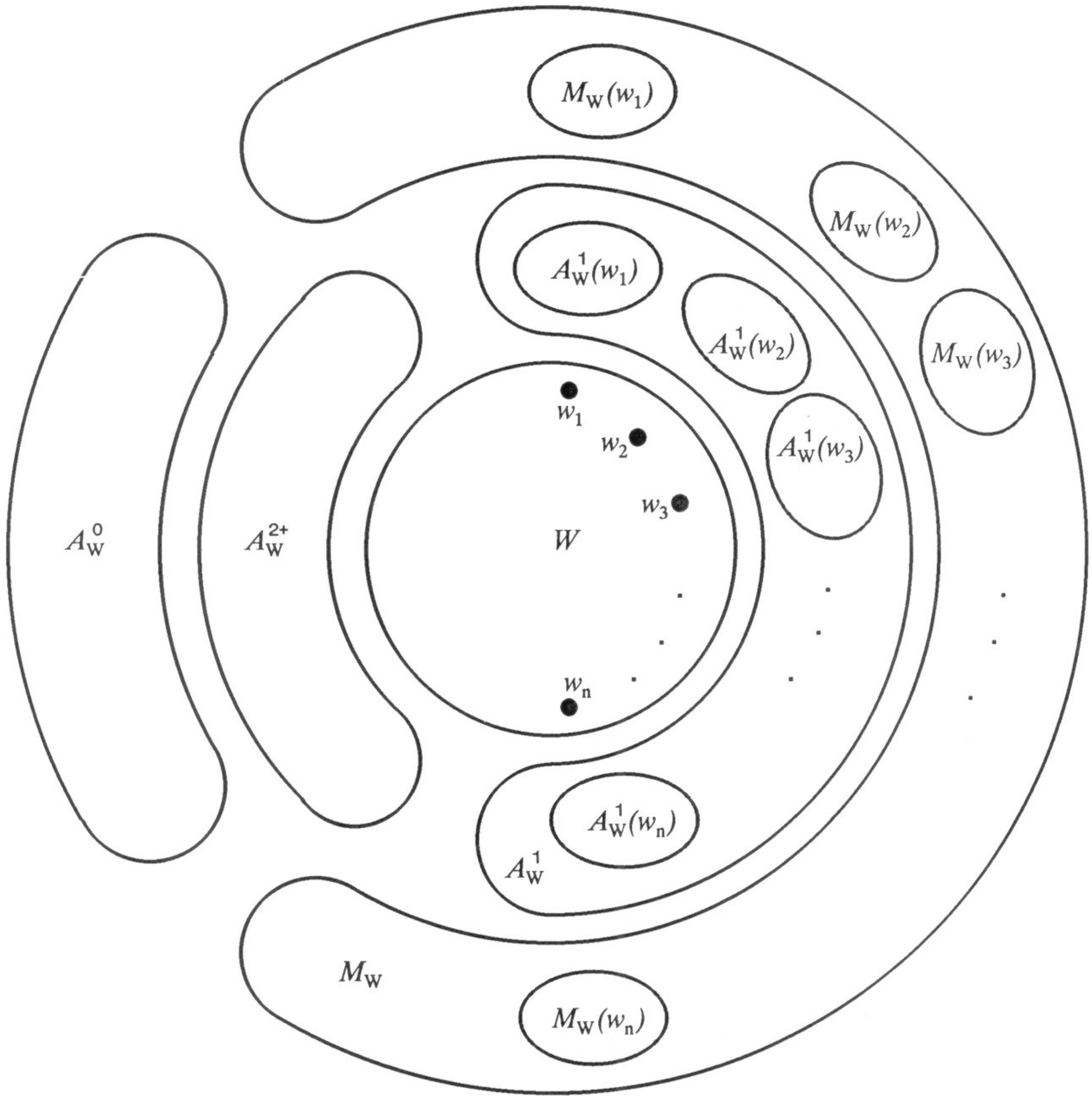
Fact 8: If G/W is 4-prime, and w_1 and w_2 are distinct vertices in W, then $(M_W(w_1) \cup A_W^1(w_1)) \cap (M_W(w_2) \cup A_W^1(w_2)) = \emptyset$.

Since 4-prime graphs contain no articulation points, $M_W \cap A_W^1$ is empty. Hence

Fact 9: if G/W is 4-prime, then A_W^0, A_W^{2+}, $M_W(w)$, $A_W^1(w)$ for all $w \in W$ form a partition of $V-W$ (Fig. 2.7).

Similarly,

Fact 10: if G/W is 3-prime, then A_W^0, A_W^∞, A_W^{1+}, $M_W(w)$ for all $w \in W$ form a partition of $V-W$.



The partition of $V - W$ into A^0_W, A^1_W, A^{2+}_W, and M_W

Fig. 2.7

We are now ready to introduce the central lemma:

LEMMA 1. If $G = (V, E)$ and if G/W is a 4-prime induced subgraph of G, then:

(1) If there exists $a \in A_W^{2+}$ then with $W_1 = W \cup \{a\}$, G/W_1 is 4-prime.

(2) If there exists $w \in W$ such that $M_W(w) \cup A_W^1(w) \neq \varnothing$ then:

(i) If there exists $m \in M_W(w)$ and $x \in (M_W - M_W(w)) \cup A_W^{2+}$ such that $(m, x) \in E$ iff $(w, x) \notin E$ (Condition i), then with $W_1 = W \cup \{m, x\}$, G/W_1 is 4-prime.

(ii) If there exists $m \in M_W(w) \cup A_W^1(w)$ and $x \in V - W - A_W^0 - M_W(w) - A_W^1(w)$ such that $\{m, x\} \cap A_W^1 \neq \varnothing$ and $(m, x) \in E$ (Condition ii), then with $W_1 = W \cup \{m, x\}$, G/W_1 is 4-prime.

(iii) If there exists $m \in M_W(w) \cup A_W^1(w)$ and $x \in V - W - A_W^0 - M_W(w) - A_W^1(w)$ such that $G/\{m = p_0, p_1, \cdots, p_{k+1} = x\}$ is either a C_{k+2} or P_{k+2} where $k \geq 1$, $p_j \in A_W^0$ for $j \in \{1, 2, \cdots, k\}$, and p_j is adjacent to p_{j+1} for $j \in \{0, 1, \cdots, k\}$ (Condition iii), then with $W_1 = W \cup \{v, p_1, \cdots, p_k, x\}$, G/W_1 is 4-prime.

(iv) If there is no m and x such that Condition (i), (ii), or (iii) is satisfied, then G is not 4-prime.

In English this means that if G/W is 4-prime, then we may attempt to augment W, creating W_1, by means of any one of four rules. This new W_1, if we can actually apply one of the rules, will induce a 4-prime subgraph of G. This lemma also gives a condition for when G is not 4-prime. It will be convenient to work with a more restricted version of this lemma (Lemma 2) which is introduced next. In Lemma 2 G is taken to be prime, hence the need for condition (2iv) in Lemma 1 is eliminated.

LEMMA 2. If G is 4-prime, $W \subseteq V$, $4 \leq |W| < |V|$, and G/W is 4-prime then at least one of the following five holds true:

(1) There exists $a \in A_W^{2+}$. In this case let $W_1 = W \cup \{a\}$.

(2) There exist $w_1, w_2 \in W, m_1 \in M_W(w_1), m_2 \in M_W(w_2)$ such that $w_1 \neq w_2$ and $(m_1, m_2) \in E$ iff $(w_1, w_2) \notin E$. In this case let $W_1 = W \cup \{m_1, m_2\}$.

(3) There exist distinct vertices $w_1, w_2 \in W, p_0 = a_1 \in A_W^1(w_1), p_k = a_2 \in A_W^1(w_2)$, and $p_i \in A_W^0$ for $i \in \{1, 2, \cdots, k\}$ such that $G/\{a_1, p_1, p_2, \cdots, p_k, a_2\} = P_{k+2}(a_1, p_1, p_2, \cdots,$

$p_k, a_2)$. In this case let $W_1 = W \cup \{a_1, a_2, p_1, \cdots, p_k\}$.

(4) There exist $w_1, w_2 \in W, m_1 \in M_W(w_1), a_2 \in A_W^1(w_2)$ such that $w_1 \neq w_2$ and there exists a simple path, $m_1, p_1, \cdots, p_k, a_2$ with $k \geq 0$ and $p_i \in A_W^0$ for $i \in \{1,2, \cdots ,k\}$. In this case let $W_1 = W \cup \{m_1, a_2, p_1, \cdots, p_k\}$.

(5) There exist $w_1, w_2 \in W, m_1 \in M_W(w_1), m_2 \in M_W(w_2)$ such that $w_1 \neq w_2$, m_1 is adjacent to m_2 iff w_1 is adjacent to w_2, and there exists a simple path or cycle, $m_1, p_1, \cdots, p_k, m_2$ with $k \geq 1$ and $p_i \in A_W^0$ for $i \in \{1,2, \cdots ,k\}$. In this case let $W_1 = W \cup \{m_1, m_2, p_1, \cdots, p_k\}$.

Furthermore, G/W_1 is 4-prime.

This lemma says that if G is 4-prime then we will be able to apply at least one of 1 through 5 to augment W in such a fashion as to produce a larger graph G/W_1 that is still 4-prime. The way that Lemma 2 is derived from Lemma 1 is that Condition (1) of Lemma 1 becomes condition (1) of Lemma 2. Condition (2i) of Lemma 1 becomes condition (2) of Lemma 2. Condition (2ii) of Lemma 1 becomes conditions (3) and (4) of Lemma 2, with $k = 0$ in each of (3) and (4). Condition (2iii) of Lemma 1 becomes condition (3), (4), and (5) of Lemma 2, with $k \geq 1$ in each of (3), (4), and (5). While Lemma 2 is not as compact, the types of augmenations which may be made to W to still yield a 4-prime graph are clearer. By utilizing the correspondence between 3-prime and 4-prime, we may immediately arrive at a corresponding lemma for 3-prime graphs:

LEMMA 3. If G is 3-prime, W is a proper subset of V, and G/W is 3-prime, then at least one of the following four holds true:

(1) There exists $a \in A_W^{1+}$. In this case let $W_1 = W \cup \{a\}$.

(2) There exists $w_1, w_2 \in W, m_1 \in M_W(w_1), m_2 \in M_W(w_2)$ such that $w_1 \neq w_2$ and $(m_1, m_2) \in E$ iff $(w_1, w_2) \notin E$. In this case let $W_1 = W \cup \{m_1, m_2\}$.

(3) There exists $w_1 \in W, m_1 \in M_W(w_1), m_2 \in A_W^{\infty}$ such that $(m_1, m_2) \notin E$. In this case let $W_1 = W \cup \{m_1, m_2\}$.

(4) There exists $w_1 \in W, m_1 \in M_W(w_1), a \in A_W^0$ such that $(m_1, a) \in E$. In this case let $W_1 = W \cup \{m_1, a\}$.

In addition, G/W_1 is 3-prime.

This lemma says that assuming G is 3-prime, we will be able to apply at least one of (1) through (4) to augment W so as to produce a larger set W_1 whose induced subgraph, G/W_1, is still 3-prime. To see how Lemma 3 follows from Lemma 2, we let $G = (V, E)$ be a 3-prime graph and z a vertex not in V such that z is adjacent to each vertex in V. Then $G' = G/(V \cup \{z\})$ is 4-prime so that Lemma 2 applies to it. Now, for each $W \subseteq V$ which induces a 3-prime subgraph of G we form a corresponding set: $W' = W \cup \{z\}$. Note that $A^0_{W'} = \varnothing$. This means that (3) and (5) in Lemma 2 are inapplicable. Thus, the form of Lemma 2 may be simplified to yield Lemma 3 in the following fashion. A^{2+}_W in Lemma 2 becomes A^{1+}_W in Lemma 3, so that (1) in Lemma 2 becomes (1) in Lemma 3. (2) in Lemma 2 becomes (2) and (3) (the second case arises since those vertices W'*–similar* to z must be accounted for) in Lemma 3. Finally, (4) in Lemma 2 becomes (4) in Lemma 3.

Proof of Lemma 1.

We first note that since G is connected if $W \neq V$ then $A_W^{2+} \cup M_W \cup A_W^1 \neq \emptyset$.

To show that (1) holds: Assume for a contradiction that $a \in A_W^{2+}$ but G/W_1 is not 4-prime. Then there is some 4-factoring partition (H_0, H_1, H_2, H_3) of W_1. Without loss of generality $a \in H_0 \cup H_1$ which means that there is exactly one element $w \in W \cap (H_0 \cup H_1)$ since $(H_0-\{a\}, H_1-\{a\}, H_2, H_3)$ is not a 4-factoring partition of W. Now $w \in H_1$ since G/W is connected. But $a \notin H_0$ (since $a \notin A_W^1$), and $a \notin H_1$ (since $a \notin M_W(w)$) so that we have a contradiction.

To show (2), assume $w \in W$ such that $M_W(w) \cup A_W^1(w) \neq \emptyset$.

Case 1: Condition i or ii is satisfied:

Assume that G/W_1 is not 4-prime so there is some 4-factoring partition (H_0, H_1, H_2, H_3) of W_1.

Case 1.1: Both m and x are elements of $H_0 \cup H_1$.

Then there is at most one element of W in $H_0 \cup H_1$ otherwise $(H_0 - \{m, x\}, H_1 - \{m, x\}, H_2, H_3)$ would be a 4-factoring partition of W. Thus, no element of W is in H_0 since W is connected. If no element of W is in H_1 then neither m nor x could lie in H_0 since neither is in A_W^0, but neither can they both lie in H_1 since then they would be W-similar. Thus, suppose $y \in W$ is an element of H_1 which means that $y = w$. Hence, both m and x would be elements of $M_W(w) \cup A_W^1(w)$ a contradiction.

Case 1.2: $m \in H_0 \cup H_1$ and $x \in H_2 \cup H_3$.

Since $(H_0 - \{m\}, H_1 - \{m\}, H_2 - \{x\}, H_3 - \{x\})$ does not 4-factor W, either $H_0 \cup H_1 - \{m\}$ or $H_2 \cup H_3 - \{x\}$ contains exactly one vertex, $y \in W$.

Case 1.2.1: $y \in H_0 \cup H_1$ and $W - \{y\} \subseteq H_2 \cup H_3$.

Then $y \in H_1$ since W is connected, and $m \in H_1$ since $m \notin A^1_{(W \cup \{x\})}$ which means that $y = w$ and $m \in M_W(w)$. But then $x \notin H_2 \cup H_3$ since x is adjacent to exactly one of m and w which is a contradiction.

Case 1.2.2: $W - \{y\} \subseteq H_0 \cup H_1$ and $y \in H_2 \cup H_3$.

Then $y \in H_2$ since W is connected, and $x \in H_2$ since $x \notin A^1_{(W \cup \{m\})}$ which means that x is W-similar to y, a contradiction.

Case 2: Condition iii is satisfied:

Assume that G/W_1 is not 4-prime so that there is some 4-factoring partition (H_0, H_1, H_2, H_3) of W_1. Since G/W has no articulation points and $\{m, x\} \subseteq M_W(w) \cup A^1_W(w)$, for each $v \in W$ there is a path in G/W from m to v which is vertex disjoint from a path from v to x. Thus, there are two vertex disjoint (in G/W_1) paths to each vertex in $W_1 - W$ for each $v \in W$. By the same token, there are also two vertex disjoint paths between each pair of vertices in $W_1 - W$. Hence G/W_1 contains no articulation points so that neither H_1 nor H_2 may contain exactly one element.

Case 2.1: $p_j \in H_1$ for some $j \in \{1, 2, \cdots, k\}$.

Then no member of W is in H_2 since only p_j's neighbors may be in H_2; hence $H_2 = N_{W_1}(p_j)$.

Case 2.1.1: W is contained solely in $H_0 \cup H_1$.

If $k = 1$, then $H_2 = \{m, x\}$, but this is a contradiction since m and x are not W-similar. If $k > 1$ then ak least one of p_j's neighbors is an element of $\{p_1, p_2, \cdots, p_k\}$ which precludes an element of W from being a member of H_1 (Fig. 2.8). This, in turn, precludes m and x from being members of $H_2 \cup H_3$. Hence $H_2 \cup H_3 \subseteq \{p_1, p_2, \cdots, p_k\}$. But since no two vertices in $W_1 - W - \{m, x\}$ have a common neighborhood of more than one vertex, $|H_1| = 1$, a contradiction.

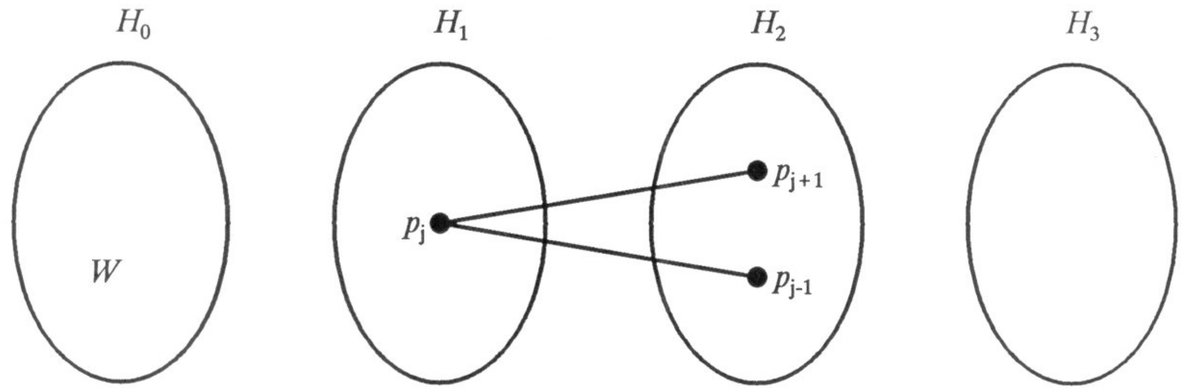


Illustration for Case 2.1.1 of Lemma 1

Fig 2.8

Case 2.1.2: W is contained solely in H_3.

Then neither m nor x is contained in $H_0 \cup H_1$ since both are adjacent to some element of W. But no two elements of $W_1 - W - \{m, x\}$ share two common neighbors so that $|H_1| = 1$, a contradiction.

Case 2.2: no p_j is in H_1 (nor in H_2) for $j \in \{1,2, \cdots, k\}$.

Then without loss of generality $p_j \in H_0$ for each $j \in \{1,2, \cdots, k\}$ so that $H_2 \cup H_3$ contains at least two elements of W. Now, $H_2 \cup H_3$ may not contain all the vertices in W since then both m and x would be in H_1 where they would be W-similar. But if $H_0 \cup H_1$ contain exactly one vertex y from W, then we would have both m and x elements of $M_W(y) \cup A_W^1(y)$, a contradiction.

Thus, we have shown that i, ii, and iii are true so that we only have to show iv. To this end, suppose that none of Conditions i, ii, or iii are met. Then we proceed to show that (H_0, H_1, H_2, H_3) 4-factors G where the H_i are defined as follows:

$H_0 = A_W^1(w) \cup \{v \in A_W^0$: there is a path from $A_W^1(w) \cup M_W(w)$ to v contained solely in $A_W^0\}$

$H_1 = M_W(w) \cup \{w\}$

$$H_2 = N_W(w) \cup N_{A_W^{2+}}(w) \cup \bigcup_{x \in N_W(w)} M_W(x)$$

$$H_3 = V - H_1 - H_2 - H_3$$

Since $M_W(w) \cup A_W^1(w) \neq \varnothing$, we have that $|H_0 \cup H_1| \geq 2$, and since $|N_W(w)| \geq 2$ we have that $|H_2 \cup H_3| \geq 2$. It is clear that every element of H_1 is adjacent to every element of H_2 since Condition 2i does not hold. It is clear that every element of A_W^0 in H_0 is adjacent to no vertex of H_2 since Condition 2iii does not hold. It is clear that every element of $A_W^1(w)$ in H_0 is adjacent to no vertex of H_2 since Condition 2ii does not hold. Thus, $(H_0 \times H_2) \cap E = \varnothing$.

By the definition of H_0, no vertex in H_1 is adjacent to a vertex in $A_W^0 \cap H_3$. Since condition 2ii does not hold, there is no vertex of H_1 adjacent to a vertex in $A_W^1 - A_W^1(w) \subseteq H_3$. Since condition 2i does not hold, if a vertex in $A_W^{2+} \cup M_W - M_W(w)$ is not adjacent to w, then it is adjacent to no member of $M_W(w)$ so no vertex in H_1 is adjacent to a vertex in $(A_W^{2+} - N_{A_W^{2+}}(w))$ $\cup$ $(MW - M_W(w) - \bigcup_{x \in N_W(w)}) \subseteq H_3$. Finally, we have that no element in H_1 is adjacent to an element in H_3 since

$$H_3 = (A_W^0 - H_0) \cup (A_W^1 - A_W^1(w)) \cup (A_W^{2+} - N_{A_W^{2+}}(w)) \cup (M_W - M_W(w) - \bigcup_{x \in N_W(w)} M_W(x)) \cup (W - N_W(w) - \{w\}).$$

Now, no vertex in $A_W^0 \cap H_3$ is adjacent to any vertex in H_0 by the definition of H_0. In fact since Condition 2iii does not hold, no vertex in $A_W^0 \cap H_0$ is adjacent to a vertex in $V - M_W(w) - A_W^1(w) - A_W^0$. Thus we must only account for the adjacencies of $A_W^1(w)$. But, since Condition 2ii does not hold, no vertex in $A_W^1(w)$ is adjacent to a vertex in $V - M_W(w) - A_W^1(w)$. Thus, $(H_0 \times H_3) \cap E = \varnothing$.

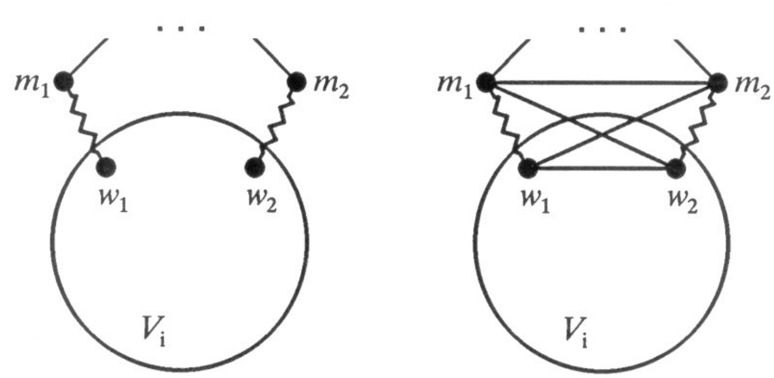
QED Lemma 1

2.5. Applications of the Central Lemma

Cor. 1 (to Lemma 2). If $G=(V,E)$ is 4-prime then there exists a maximal 4-constructing sequence $(W_1, W_2, \cdots, W_k)$ such that if $1<j\leq k$ then G/W_j is obtained from G/W_{j-1} by applying one of rules 1,2, or 3 in Lemma 2.

Proof of Cor. 1: Suppose $(V_0, V_1, \cdots, V_l)$ is a maximal constructing sequence subject to Lemma 2. What will be shown is that for each application of rule 5 there is a another maximal constructing sequence subject to Lemma 2 with one less application of rule 5 (and one more application of rule 4) --- thus, there is a maximal constructing sequence subject to Lemma 2 with no application of rule 5. It will also be shown that for each application of rule 4 there is a another maximal constructing sequence subject to Lemma 2 with one less application of rule 4 (and no more applications of rule 5) --- thus, there is a maximal constructing sequence subject to Lemma 2 with no applications of rule 4 nor of rule 5.

Suppose that G/V_{i+1} is formed from G/V_i by application of Rule 5 (Fig. 2.9). For $0\leq j\leq i$, if $w_1\notin V_j$ let V'_j be the set V_j, and otherwise ($w_1\in V_j$) let $V'_j = V_j\cup\{m_1\}-\{w_1\}$ (we swap vertex w_1 with m_1). Since G/V_j is isomorphic to G/V'_j, $(V'_0, \cdots, V'_i, V_{i+1}, \cdots, V_l)$ is a constructing sequence (though not maximal). Since m_1 is adjacent to m_2 iff w_1 is adjacent to w_2, $(V'_0, \cdots, V'_i, V_{i+1}-\{w_1\}, V_{i+1}, \cdots, V_k)$ is a maximal constructing sequence where $G/(V_{i+1}-\{w_1\})$ is formed from G/V'_i by application of rule 4 and

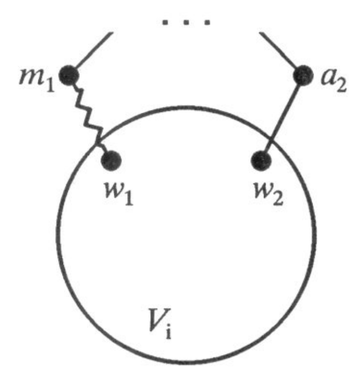


The two possible applications of rule 5 (Lemma 2)

Fig 2.9

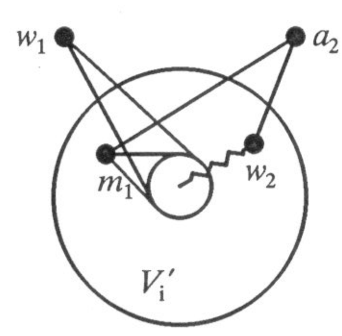
G/V_{i+1} is formed from $G/(V_{i+1}-\{w_1\})$ by application of rule 1. Thus, we have a new maximal constructing sequence subject to Lemma 4 where the number of applications of rule 5 has decreased by 1 (although the number of applications of rule 4 has increased by 1). Hence, there exists a maximal constructing sequence subject to Lemma 4 which has no applications of rule 5.

Suppose that G/V_{i+1} is formed from G/V_i by application of Rule 4 (Fig. 2.10). For $0 \le j \le i$, if $w_1 \notin V_j$ let V'_j be the set V_j, and otherwise ($w_1 \in V_j$) let $V'_j = V_j \cup \{m_1\}-\{w_1\}$. Since G/V_j is isomorphic to G/V'_j, $(V'_0, \cdots, V'_i, V_{i+1}, \cdots, V_l)$ is a constructing sequence (though not necessarily maximal). If $|V_{i+1}-V'_i| \ge 3$ (I.e. $k \ge 1$) then $(V'_0, \cdots, V'_i, V_{i+1}-\{w_1\}, V_{i+1}, \cdots, V_l)$ is a maximal constructing sequence where $G/V_{i+1}-\{w_1\}$ is formed from G/V'_i by application of rule 3 and G/V_{i+1} is formed from $G/V_{i+1}-\{w_1\}$ by application of rule 1. Thus, suppose $V_{i+1}-V'_i = \{w_1, a_2\}$ (Fig. 2.11). If $a_2 \in M_{V'_i}$, then $(V_0, \cdots, V'_i, V_{i+1}, \cdots, V_l)$ is a maximal constructing sequence where G/V_{i+1} is formed from G/V'_i by application of rule 2. If, on the other hand, $a_2 \notin M_{V'_i}$, then $a_2 \in A^{2+}{}_{V'_i}$ so $(V'_0, \cdots, V'_i, V'_i \cup \{a_2\}, V_{i+1}, \cdots, V_l)$ is a maximal constructing sequence where $G/(V'_i \cup \{a_2\})$ is formed from G/V'_i by application of rule 1 and G/V_{i+1} is likewise formed from $G/(V'_i \cup \{a_2\})$ by application of rule 1. In each case, the new maximal constructing sequence is subject to Lemma 2, but there is one fewer occurence of rule 4 being used (and the same number of applications of rule 5). **QED** Cor. 1



Application of Rule 4 (Lemma 2)

Fig 2.10



Application of Rule 4 (Lemma 2)

Fig 2.11

Cor. 2 (to Lemma 3). If $G = (V, E)$ is 3-prime then there exists a maximal 3-constructing sequence $(W_1, W_2, \cdots, W_k)$ such that if $1<j \leq k$ then G/W_j is obtained from G/W_{j-1} by applying one of rules 1 or 2 in Lemma 2.

Proof of Cor. 2: lForm $G'(V', E')$ where $V' - V = \{z\}$ and $E' = \cup \{\{z\} \times V\}$ so that G' is 4-prime. Let $(W_1, W_2, \cdots, W_k)$ be a maximal 4–constructing sequence according to Cor. 1 where $z \in W_1$ (thus, $G/W_1 = C_5^{2,3}$). Clearly, rule 3 of Lemmma 2 is not used since $N_{V'}(x) = V$.

We now show that we can form a maximal 4–constructing sequence $(W_1, W_2, \cdots, W_k = V')$ for G' where $G'/W_1 = C_5^{2,3}$ but $W_{i+1} \cap M_{W_i}(x) = \varnothing$) for $0<i<k$ (I.e. no vertices from $M_{W_i}(x) = A_{W_i^{**}}$ are ever added). As in the proof of Cor. 1, suppose $(V_0, \cdots, V_l)$ is a maximal 4–constructing sequence for G' and G'/V_{i+1} is formed from G'/V_i by application of rule 2 (of Lemma 2) and $M_{V_i}(x) \cap V_{i+1} = \{m\}$. Then $V_{i+1} - V_i = \{m, m_1\}$ where $m_1 \in M_{V_i}(w_1)$. For $0 \leq j \leq i$, if $w_1 \notin V_j$, let V'_j be the set V_j, and otherwise ($w_1 \in V_j$) let $V'_j = V_j \cup \{m_1\} - \{w_1\}$ (we swap vertex w_1 with vertex m_1). Since G'/V_j is isomorphic to G'/V'_j, $(V'_0, \cdots, V'_i, V_{i+1}, \cdots, V_l)$ is a maximal 4–constructing sequence in $m \in M_{V'_i}$, and otherwise $(V'_0, \cdots, V'_i, V'_i \cup \{m\}, V_{i+1}, \cdots, V_l)$ is a maximal 4–constructing sequence (this is the case where $m \in A_{V'^{2+}_i}$). Thus, we may easily find a maximal 4–constructing sequence $(W_1, \cdots, W_k = V')$ (and where $z \in W_1$) subject only to rules 1 and 2 of Lemma 2 such that $W_{i+1} \cap M_{W_i}(x) = \varnothing$ for $0 \leq i \leq k$. Hence $W_{i+1} \cap (M_{W_{i+1}}(x) \cap A_{W_i'}) = \varnothing$. Hence $(W_1 - \{z\}$, $W_2 - \{z\}$, $\cdots$, $W_k - \{z\} = V)$ is a maximal 3–constructing sequence for G using only rules 1 and 2 of Lemma 3. **QED** Cor. 2

As an application of the previous Lemmas, Theorem 1 and Theorem 2 have a remarkable refinement. Since 3-prime graphs contain no similar pairs, Theorem 1 tells us that such graphs contain an imbedded P_4. Theorem 3 states that not only do 3-prime graphs contain an imbedded P_4, but that all but at most one vertex is not contained in an embedded P_4! Similarly, if G is 4-

prime, then not only is there some induced subgraph of G which is isomorphic to one of the graphs in Fig. 1.6, but *each* vertex of G is contained in an induced subgraph of G which is isomorphic to one of the graphs in F_1 (Fig. 1.5). More precisely,

Theorem 3: If $G = (V, E)$ is 3-prime with $|V| \geq 3$, then (i) every vertex in V is contained in an induced P_4 or DW (see Fig. 1.4) and (ii) at most one vertex in V is not contained in an induced P_4.

Theorem 4: If $G = (V, E)$ is 4-prime with $|V| \geq 4$, then every vertex in V is contained in an induced graph of G which is isomorphic to one of those from Fig. 1.5.

Note that a DW is 3-prime and hence the theorem also applies to it. Furthermore, each of the graphs in Fig. 1.5 contains as an induced subgraph one of the graphs in Fig. 1.6, and each of the graphs in Fig. 1.6 and Fig. 1.5 are 4-prime.

Proofs of Theorems 3 and 4 The idea behind each of the proofs is induction on the size of the graph (part (ii) of Theorem 3 is proved differently: by assuming the contrary and showing a contradiction). Specifically, the statement of the theorem is taken to be the induction hypothesis for a prime graph of size i. The basis step, $i = 4$ for 3-prime graphs and $i = 5$ for 4-prime graphs is easy to see since the only 3-prime graph of size 4 is a P_4 and the only 4-prime graph of size 5 is a C_5, *tepee*, or *house*. Next, the induction hypothesis is assumed to be true for all prime graphs of size k or less, and the truth of the induction hypothesis is demonstrated for an arbitrary prime graph, $G = (V, E)$, of size $k+1$ (where $k \geq 5$ for G 3-prime and $k \geq 6$ for G 4-prime. If G is 3-prime, then Theorem 1 guarantees that G contains a 3-prime, proper, induced subgraph. If G is 4-prime and contains no 4-prime, proper, induced subgraph, then Theorem 2 guarantees that G is one of the graphs in Fig. 1.6 so that G satisfies the induction hypothesis. Thus, we need only consider 4-prime graphs which contain a proper induced 4-prime subgraph. Let W be a largest, proper subset of V such that G/W is (3 or 4)-prime. The induction hypothesis applies to G/W since W is a proper subset of V. By Cor. 1 to Lemma 2 (or Lemma 3) there are only a few ways to obtain G from G/W, and it is demonstrated that the induction hypothesis is maintained for each of them.

Before diving into the meat of the proofs, it is convenient to introduce several ancillary facts:

Fact 11. If $G = (V, E)$, $x \in W \subseteq V$, and $y \in V - W$ such that y is W-similar to x, then for every $W_1 \subseteq W$ such that $x \in W_1$, G/W_1 is isomorphic to $G/(W_1 \cup \{y\} - \{x\})$

Fact 12. $G(V, E)$ is 3-prime iff the complement of G is 3-prime. The complement of a P_4 is a P_4, and the complement of a DW is a DW.

Fact 13a. If $G(V, E)$ is a graph where four of the vertices induce a P_4 and the fifth vertex is adjacent to some but not all of the other four vertices, then the fifth vertex, together with some of the other four induces either a P_4 or DW in G.

Proof of Fact 13a. Let G be isomorphic to G_1=(V_1, E_1) where $V_1 = \{a, b, c, d, v\}$, $G_1/(V_1 - \{v\}) = P_4(a, b, c, d)$ and v is adjacent to some but not all of $V_1 - \{v\}$ (Fig. 2.12). If $N(v) = \{b, c\}$ then G is a DW. If $N(v) = \{a\}$, $\{a, d\}$, $\{c\}$, or $\{c, d\}$ then $\{v, a, b, c\}$ induces a P_4 on G_1. If $N(v) = \{d\}$, $\{a, b\}$, or $\{b\}$ then $\{v, b, c, d\}$ induces a P_4 on G_1. If $N(v) = \{b, d\}$, $\{b, c, d\}$, or $\{a, c, d\}$ then $\{v, a, b, d\}$ induces a P_4 on G_1. If $N(v) = \{a, c\}$, $\{a, b, c\}$, or $\{a, b, d\}$ then $\{v, a, c, d\}$ induces a P_4 on G_1.

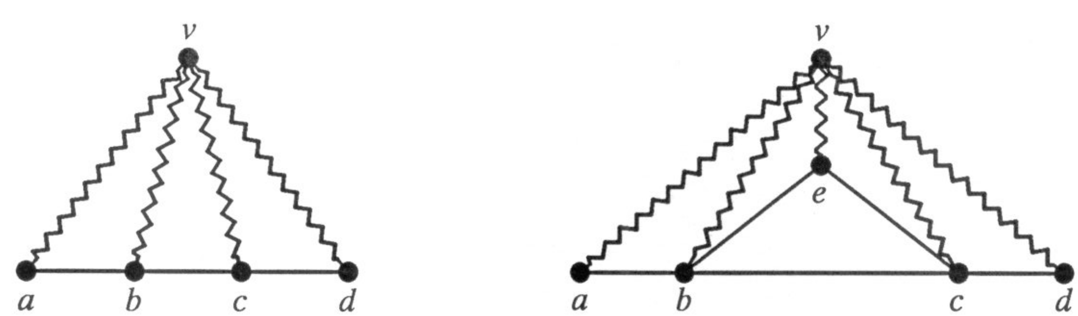


Figure for the proof of Fact 13

Fig. 2.12

Fact 13b. If $G(V,E)$ is a graph such that five of the vertices induce a DW, and the sixth vertex is adjacent to some but not all of the other five, then the sixth vertex together with some of the other five vertices induces either a DW or a P_4 on G.

Proof of Fact 13b. Let G be isomorphic to $G_1=(V_1,E_1)$ where $V_1=\{a,b,c,d,e,v\}$, $G_1/(V_1-\{v\})=DW(a,b,c,d,e)$ and v is adjacent to some but not all of $V_1-\{v\}$ (Fig. 2.12). If $N(v)=\{a,b,c,d\}$, then $\{a,v,c,e\}$ induces a P_4 on G_1. If v is adjacent to some but not all of the vertices in $\{a,b,c,d\}$ then we apply Fact 13a. If v is adjacent to none of $\{a,b,c,d\}$ then $\{v,e,b,a\}$ induces a P_4 on G_1.

Proof of Theorem 3:

To prove Theorem 3 the following Induction Hypothesis is used: Every vertex in a 3-prime graph $G=(V,E)$ of size $4 \le |V| \le i$ is contained in an induced P_4 or in an induced DW. The basis step was shown in the first paragraph of this section. The induction step follows: suppose that the induction hypothesis is true for $i \le k$. We show this implies the truth of the induction hypothesis for $i=k+1 \ge 5$. Thus, suppose $|V|=k+1$ and $(W_0, W_1, \cdots, W_j=V)$ is a maximal constructing sequence subject to Lemma 3. By our induction hypothesis, each vertex of each W_l ($l \in \{0, \cdots, j-1\}$) is contained in a P_4 or a DW (which is embedded in W_l). All we have to do is show that every vertex in the application of the last rule is also contained in a P_4 or DW. We have already taken care of $j=0$ in our basis step.

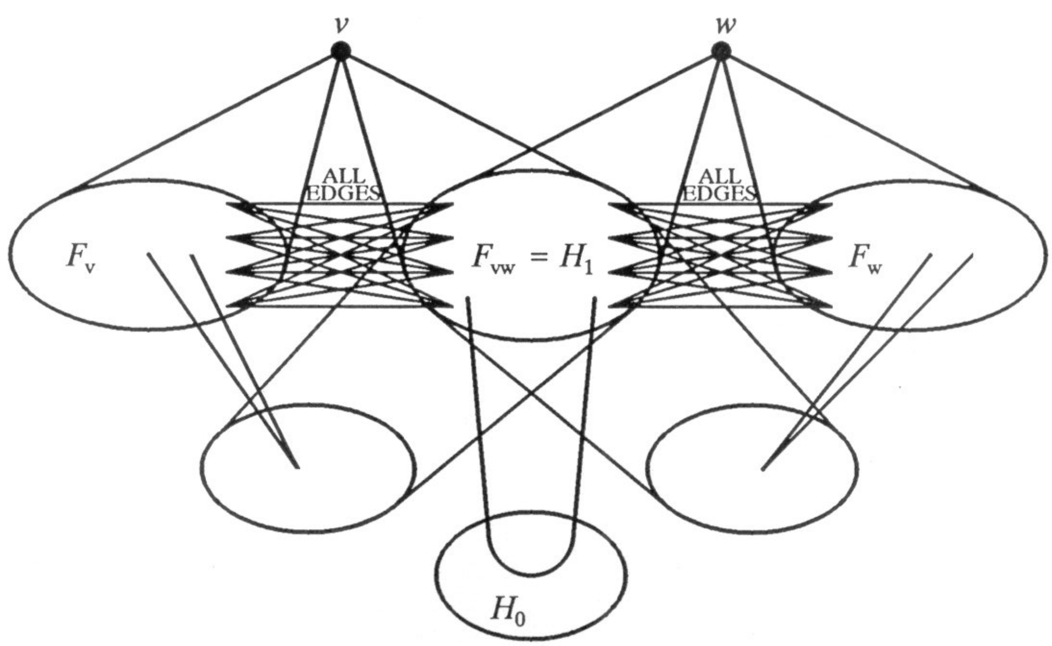
Suppose the last rule applied is rule 2. Then both the added vertices are W_{j-1}-similar to some vertices in W_{j-1} so we apply Fact 11. If the last rule applied is rule 4 then since w_1 must be the endpoint of a P_3 (since each vertex in a P_4 is the endpoint of a P_3) embedded in G/W_{j-1}, m_1 will also be the endpoint of a P_3 embedded in $G/(W_{j-1} \cup \{m_1\})$. But since a is adjacent only to m_1, it is the endpoint of a P_4 containing both a and m_1.

If we use rule 3 last, then we examine the complement of $G=G/W_j$ and G/W_{j-1} which are both 3-prime by Fact 12. In fact, in the complemented graph, a and m_1 were added by using rule

2 so they are thus in a P_4 in the complement of G/W_j. Thus, since the complement of a P_4 is a P_4, they are also in a P_4 in G/W_j

Suppose we use rule 1, adding a. If a is adjacent to some but not all the elements of W_0 then we apply Fact 13. Thus, suppose that a is adjacent no vertex in W_0 (just take complements to treat the case of a adjacent to each vertex of W_0). Then let l be the smallest integer such that a is adjacent to some vertex in W_l but adjacent to no vertex in W_{l-1} (such a vertex exists since G is connected). By our inductive hypothesis each vertex of W_l is contained in some P_4 or DW embedded within W_l. Thus, if $y \in N_{W_l}(a)$ then y is embedded in an induced subgraph of G/W_l which is a DW or P_4 but at least two of whose vertices are not adjacent to a (since $1 \leq |N_{W_l}(a)| \leq 2$). Hence, Fact 13 applies.

To prove the second half of Theorem 3 we need to show that we cannot have two vertices in a 3-prime graph, $G = (V, E)$, not contained in an embedded P_4. Thus, assume for a contradiction that neither $v, w \in V$ are contained in induced P_4s of G. Thus, they are both contained in embedded DWs and without loss of generality assume that $(v, w) \notin E$ (else just complement the



Decomposing G in showing (2) of Theorem 3

Fig. 2.13

graph). Then form (see Fig. 2.13):

$$F_v = \{x \in V : (x,v) \in E \text{ and } (x,w) \notin E\},$$
$$F_w = \{x \in V : (x,v) \notin E \text{ and } (x,w) \in E\},$$
$$F_{vw} = \{x \in V : (x,v) \in E \text{ and } (x,w) \in E\},$$
$$\text{with} \quad F = F_v \cup F_w \cup F_{vw}$$

Suppose $a \in F_v, b \in F_w$, and $c \in F_{vw}$. Then $(a,b) \notin E$ (else $\{v,a,b,w\}$ induces a P_4), $(a,c) \in E$ (else $\{a,v,c,w\}$ induces a P_4), and $(b,c) \in E$ (else $\{b,w,c,v\}$ induces a P_4). It is clear that $F_{vw} \neq \varnothing$ else the shortest path from v to w would yield a P_4. It is also clear that $F_v \cup F_w \neq \varnothing$ since v and w are not similar. Further, every vertex in $V - F - \{v,w\}$ must be adjacent to some vertex in F since otherwise the shortest path from it to v will yield a P_4 that v is in.

Every vertex $d \in V - F$ which is adjacent to some $a \in F_v \cup F_w$ is adjacent to every vertex $b \in F_{vw}$ or else either $\{d,a,b,w\}$ or $\{d,a,b,v\}$ would induce a P_4. No vertex $e \in V - F$ is adjacent to $a \in F_v$ and $c \in F_w$ or else $\{v,a,e,c\}$ would induce a P_4. Suppose $g \in V - F$ is adjacent to no element of $F_v \cup F_w$ and $h \in V - F$ is adjacent to some $z \in F_v \cup F_w$. Then g is not adjacent to h or else either $\{g,h,z,v\}$ or $\{g,h,z,w\}$ would induce a P_4 (in the case where $\{g,h\} \cap \{v,w\} = \varnothing$. The other case is taken care of by the definition of F.).

Finally, it is easily verified that with the above relations, we will obtain a 3-decomposing partition (H_0, H_1, H_2) as follows:

$H_0 = \{x \in V - F - \{v,w\} : x \text{ is adjacent to no element of } F_v \cup F_w\}$

$H_1 = F_{vw}$

$H_2 = V - H_1 - H_2 = F_v \cup F_w \cup \{v,w\} \cup N(F_v \cup F_w) - F_{vw}$

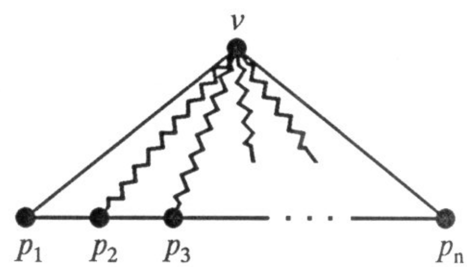
It is easy to see that $|H_0 \cup H_1| \geq 1, |H_2| \geq 2, H_2 \times H_1 \subseteq E, H_0 \times H_2 \cap E = \varnothing$ which is a contradiction since G/V is assumed 3-prime.

QED Theorem 3

Proof of Theorem 4 (from page 44)

Before commencing with the main proof, it is useful to introduce a few ancillary facts.

Fact 14. Suppose $G = (V, E)$, $G/\{p_1, p_2, \cdots, p_n\} = P_n(p_1, p_2, \cdots, p_n)$ where $n \geq 4$, v is adjacent to p_1 and to p_n, and k is the smallest integer greater than 3 such that v is adjacent to p_k. Then $R = \{v, p_1, p_2, \cdots, p_k\}$ induces one of the graphs of Fig. 1.5 (see Fig. 2.14).



Used in proving Fact 14

Fig. 2.14

Proof of Fact 14. If v is adjacent to neither p_2 nor p_3, then R induces $C_{k+1}(p_1, p_2, \cdots, p_k, v)$. If v is adjacent to p_2 but not to p_3, then R induces $C_{k+1}^{2}(p_1, p_2, \cdots, p_k, v)$. If v is adjacent to p_3 but not to p_2 then R induces $C_{k+1}^{3}(p_1, p_2, \cdots, p_k, v)$. Finally, If v is adjacent to both p_2 and p_3 then R induces $C_{k+1}^{2,3}(p_1, p_2, \cdots, p_k, v)$.

Fact 15. If $G = (V, E)$ is a graph with no articulation points, and $v \in V$ is a vertex with exactly two neighbors which are not V-similar, then v is embedded in a graph isomorphic to one of those of Fig. 1.5.

Proof of Fact 15. There exists a vertex p_2 which dissimilates v's neighbors, p_1 and x from each other. Assume without loss of generality that p_2 is adjacent to p_1 and not to x. Now, let $p_2, p_3, \cdots, p_k, x$ be the shortest path from p_2 to x not passing through p_1 so that $k \geq 3$. Then $v, x, p_k, p_{k-1}, \cdots, p_2$ is a simple path of length $k+1$ so we appeal to Fact 14 with v one endpoint of our simple path and p_1 the apex.

Fact 16. Given $G(V,E)$ and $W \subseteq V$ with G/W isomorphic to one of the graphs of Fig. 1.5. If $v \in V-W$ such that $|N_W(v)| \geq 2$ then v, together with some subset of W induces a subgraph in G isomorphic to one of those in Fig. 1.5.

Proof of Fact 16. We show this by proving each of the following subfacts. Suppose $W \subseteq V$ such that $V-W = \{v\}$ and $|N_W(v)| \geq 2$. Then there exists a subset of V, X, such that $v \in X$ and G/X is isomorphic to one of the graphs in Fig. 1.5 whenever G/W is:

Fact 16a. $C_n(p_1, p_2, \cdots, p_n)$ with $n \geq 5$.
Fact 16b. $C_n^2(p_1, p_2, \cdots, p_n)$ with $n \geq 5$.
Fact 16c. $C_n^3(p_1, p_2, \cdots, p_n)$ with $n \geq 5$.
Fact 16d. $C_n^{2,3}(p_1, p_2, \cdots, p_n)$ with $n \geq 5$.
Fact 16e. $MTP(a, b, c, d, e, f)$

If $v \in M(w)$ for some $w \in W$, then it is easy to see that $G/(W \cup \{v\} - \{w\})$ is isomorphic to G/W (Fact 11). Further, if $N_W(v)$ has exactly two elements, then Fact 15 assures us that v is in an induced subgraph isomorphic to one of those in Fig. 1.5. So, for the remainder of this proof, suppose v is not W-similar to any vertex of W and is adjacent to at least three elements of W. We now show each of the 5 subfacts in turn.

Remainder of proof of Fact 16a:

Any two non-adjacent vertices in a C_n with $n \geq 5$ are contained in a simple path (of at least four vertices) embedded in the C_n. Since v is adjacent to 3 or more vertices of W then at least two of these vertices are not adjacent. Hence, we may apply Fact 14.

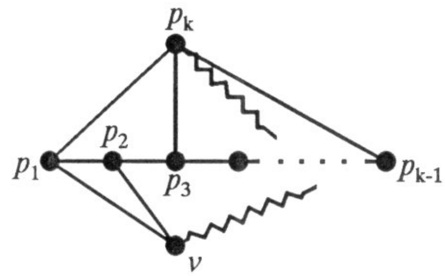
Remainder of proof of Fact 16b:

If $n \geq 6$ then $\{p_2, p_3, \cdots, p_n\}$ induces a C_{n-1} at least two of whose vertices v is adjacent to so that Fact 16a applies. Thus, suppose that $n = 5$. If $N(v) = \{p_1, p_4, p_5\}$, $\{p_1, p_3, p_4, p_5\}$, $\{p_2, p_3, p_4, p_5\}$, or $\{p_2, p_4, p_5\}$ then $\{p_1, p_2, p_4, p_5, v\}$ induces a *tepee*. If $N(v) = \{p_1, p_2, p_3\}$, $\{p_1, p_2, p_3, p_4\}$, or $\{p_2, p_3, p_5\}$ then $\{p_1, p_2, p_3, p_5, v\}$ induces a *tepee*. If $N(v) = \{p_1,$

$p_3, p_4\}$ then $\{p_1, p_2, p_3, p_5, v\}$ induces *house* (p_5, p_1, v, p_3, p_2). Finally, if $N(v) = \{p_1, p_2, p_3, p_4, p_5\}$ then $\{p_1, p_2, p_3, p_4, v\}$ induces *tepee* (p_1, p_2, p_3, p_4, v).

Remainder of proof of Fact 16c:

Case c.1 $n \geq 7$ (see Fig. 2.15)

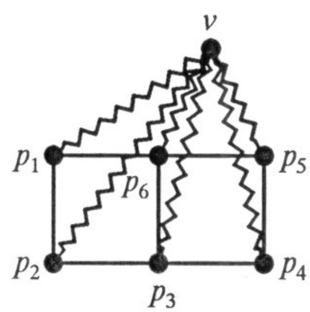


Case c.1. v is adjacent neither to p_3 nor to p_k.

Fig. 2.15

Since $\{p_3, p_4, \cdots, p_n\}$ induces a C_{n-2}, if v is adjacent to two or more of these vertices, we are done by Fact 16a. Thus, suppose that v is adjacent to p_1, p_2, and exactly one other vertex. This third vertex is neither p_3 nor p_n (since v is not in M_W) so $\{v, p_1, p_2, p_3, p_n\}$ induces *house* (v, p_2, p_3, p_n, p_1).

Case c.2 $n = 6$ (see Fig. 2.16)



v is adjacent to at least three vertices, but is similar to no p_i.

Fig. 2.16

If v is adjacent to at least five vertices then four of v's neighbors induce a P_4 so that v, together with them, induces a *tepee*. Thus, we have two cases left:

Case c.2.1: v is adjacent to exactly four vertices.

If v is adjacent to neither p_3 nor to p_6, then $\{v, p_1, p_2, p_3, p_6\}$ induces *house* (v, p_2, p_3, p_6, p_1). Thus, without loss of generality, suppose v is not adjacent to p_1. If v is also not adjacent to p_2 then $\{p_1, p_2, p_3, p_6, v\}$ induces a *house*. If v is adjacent to neither p_1 nor p_3, then $\{p_2, p_3, p_4, p_5, v\}$ induces a *house*. If v is adjacent to neither p_1 nor p_5 then $\{p_1, p_2, p_4, p_5, p_6, v\}$ induces a *setp−up*. Finally, if v is not adjacent to p_1 nor to one of either p_4 or p_6 then $\{v\} \cup N(v)$ induces a *tepee*.

Case c.2.2: v is adjacent to exactly three vertices.

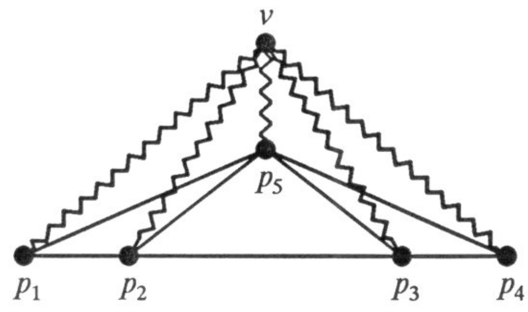
If v is adjacent to neither p_3 nor to p_6, then two of v's neighbors are adjacent, say v_j and v_{j+1}. In this event, $\{v, p_3, p_6, v_j, v_{j+1}\}$ induces a *house*. If v is adjacent to both p_3 and p_6, then designate by v_j and v_{j+1} the adjacent vertices v is not adjacent to. In this event $\{v, p_3, p_6, v_j, v_{j+1}\}$ induces a *house*. Thus without loss of generality suppose v is adjacent to p_3 and not to p_6. Then v is not adjacent to both p_1 and p_5, both p_2 and p_1, or both p_4 and p_5 since it is not similar to p_6, p_2, and p_4, respectively. Hence, we must have v adjacent to p_2 (in which case $\{v, p_1, p_2, p_3, p_6\}$ induces a *house*) or v adjacent to p_4 (in which case $\{v, p_3, p_4, p_5, p_6\}$ induces a *house*).

Case c.3: $n = 5$

This is the same as $n = 5$ in Fact 16b.

Remainder of proof of Fact 16d:

If $n \geq 6$ then $\{p_2, p_3, \cdots p_n\}$ induces $C_{n-1}^2(p_2, p_2, \cdots, p_n)$ so that Fact 16b applies. Thus, suppose that $n = 5$ (see Fig. 2.17). If v is adjacent to both p_1 and p_4 then Fact 14 applies. If v is adjacent to neither p_1 nor p_4 then $\{p_1, p_2, p_3, p_4, p_5, v\}$ induces an *MTP*. If v is adjacent to exactly one of p_1 and p_4, then (since $v \notin M_W$) v is adjacent to p_2 and p_3 but not to p_5. Hence $\{v\} \cup (W - (N(v) - \{p_2, p_3\}))$ induces a *tepee*.



v is adjacent to at least three vertices, but is similar to none of them.

Fig. 2.17

Remainder of proof of Fact 16e:

Since $\{a, b, c, d, e\}$ induces $tepee(a, b, c, d, e)$ and v is adjacent to at least two of it's vertices, Fact 16d applies.

QED Fact 16

To prove Theorem 4 the following Induction Hypothesis is used: Every vertex in a 4-prime graph $G = (V, E)$ of size $|V| \leq i$ is contained in an induced subgraph of G which is isomorphic to one of the graphs in Fig. 1.5. The basis step was shown in the first paragraph of this section. The induction step follows: suppose that the induction hypothesis is true for $i \leq k$. We show this implies the truth of the induction hypothesis for $i = k+1 \geq 5$. Thus, suppose $|V| = k+1$ and $(W_0, W_1, \cdots, W_j = V)$ is a constructing sequence subject to Lemma 2. By our inductive hypothesis, each vertex of each W_l ($l \in \{0, \cdots, j-1\}$) is embedded in a graph isomorphic to one of those in Fig. 1.5. All we have to do is show that every vertex in the final application of a rule from Lemma 2 is also embedded in one of the graphs of Fig. 1.5. If $j = 0$ (we never apply a rule), then G is isomorphic to one of the graphs in Fig. 1.6, each of which is also one of the graphs in Fig. 1.5 so each vertex in G trivially satisfies the hypothesis. For the remainder, we take $j \geq 1$.

Suppose the final rule applied is rule 2. Then both the added vertices are W_{j-1}-similar to some vertices in W_{j-1} so we apply Fact 11. If, on the final step, we add some elements of $A^0_{W_{j-1}}$, then each of those vertices has degree 2 so that they and all the vertices they are adjacent to are contained in some graph isomorphic to one of those in Fig. 1.5. by Fact 15. If on the last step we add some elements of $A^1_{W_{j-1}}$, then the degree of each of these vertices is 2 and we apply Fact 15. Thus, we have accounted for applying each of steps 2, 3, 4, or 5.

Now, suppose that in the final step we apply rule 1 on v. That is, $v \in A^{2+}_{W_{j-1}}$, and $W_j - W_{j-1} = \{v\}$. Then, if $j = 1$ we are done by Fact 14 and if $\deg(v) = 2$ we are done by Fact 15. So suppose $\deg(v) \geq 3$ and $j \geq 2$ and now consider W_{j-2}. If $v \in M_{W_{j-2}}$ then we are done by Lemma 2. If $v \in A^{2+}_{W_{j-2}}$ then we could have added v at that point to get a graph which had size less than $k+1$, was 4-prime and contained v; namely $G/(W_{j-2} \cup \{v\})$ so that the inductive hypothesis would have applied to v. Thus, we need only consider the case where $|N_{W_{j-2}}(v)| \leq 1$.

If v was in $A^0_{W_{j-2}}$ then neither rule 1 nor 2 was applied (to W_{j-2} to obtain W_{j-1}) since $\deg(v) \geq 3$. However, since all of the vertices added to W_{j-2} to obtain W_{j-1} according to either rule 3,4,or 5 are in one common induced subgraph of G/W_{j-1} isomorphic to one of those in Fig. 1.5, we may apply Fact 16 here since $\deg(v) \geq 3$.

Thus, we may suppose that v was in $A^1_{W_{j-2}}$. If one of steps 3, 4, or 5 was applied to W_{j-2} to obtain W_{j-1} then there is some $Y \subseteq W_{j-1}$ such that $W_{j-1} - W_{j-2} \subseteq Y$ and G/Y is isomorphic to one of the graphs in Fig. 5 (I.e. there is an induced subgraph of G/W_1 containing all the vertices in $W_{j-2} - W_{j-1}$ which is isomorphic to one of those in Fig. 1.6). Since v is adjacent to at least two vertices in $W_{j-1} - W_{j-2}$ we can apply Fact 16. We could not have applied rule 1 by our assumption that $\deg(v) \geq 3$. Thus, suppose rule 2 was applied to some $m_1 \in M_{W_{j-2}}(w_1)$, $m_2 \in M_{W_{j-2}}(w_2)$, $w_1 \neq w_2$. Then, since $\deg(v) \geq 3$ we have that (v, m_1), $(v, m_2) \in E$. Finally, since the one neighbor of v in W_{j-2} can not be both w_1 and w_2, we could have applied rule 4 with v and one of m_1 or m_2.

QED Theorem 4

2.6. Implementations

Two interesting questions which may be asked about prime graphs are the following: Given that $G=(V,E)$ is 4-prime and that $W \subseteq V$ induces a 4-prime graph on G, is there an efficient algorithm to calculate the various associated sets (M_W, A_W^1, A_W^∞, and A_W^{2+}), and is there an efficient algorithm to calculate which rule of Lemma 2 (hence Lemma 1) may then be applied? The answer to both questions is yes, and an algorithm is presented for the solution to each, together taking $O(|V| \times |E|)$ time. In fact, the algorithms may be applied incrementally, and the time over all the invocations is still $O(|V| \times |E|)$. In other words, even though the algorithm takes $O(|V| \times |E|)$ time for a single application, it will also take $O(|V| \times |E|)$ time when iterated over all the vertices in the graph (where each time between passes W is agumented according to the rule that the algorithm determined on the previous pass).

In describing the algorithm(s) to solve the problem, it will be convenient to use the following notation (they may be thought of as either abstract objects or data structures (in which case all operations dealing with them take $O(1)$ time). Suppose $G=(V,E)$, $W \subseteq V$, and $v \in V$. Define C(ommon)N(eighborhood)$Set_W(v) = \{x \in V - W \mid N_W(x) = N_W(v)\}$. $CNSet_W(v)$ is the subset of V composed of those elements of V whose neighborhoods in W are identical to that of v. Note that $CNSet_\varnothing(v) = V$. Define $CNSet_W^+(v) = \{x \in V - W \mid N_W(x) = N_W(v) \cup \{v\}\}$. $CNSet_W^+(v)$ is the subset of V composed of those elements of $V-W$ adjacent to v whose neighborhoods in $W-\{v\}$ are identical to that of v. Note that $CNSet_W(v) \cap CNSet_W^+(v) = \varnothing$, and if $v \in W$ then $CNSet_W(v) \cup CNSet_W^+(v) = M_W(v)$. The reason for this definition is to split up $M_W(v)$ into two sets, one in which each of the vertices is adjacent to v and the other in which none of the vertices is adjacent to v.

The algorithm is presented in two parts. The first part (algorithm B1) takes as input $G=(V,E)$, $W \subseteq V$, and $w \in V - W$. The algorithm also assumes that $CNSet_W(x)$, $CNSet_W^+(x)$, and $A_W^1(x)$ for each $x \in V$ have been determined and that A_W^0 and A_W^{2+} are also known.

```
A0.Child ← CreateNewEmptyCNSet
A0.Child.CurrentVertex ← w
w.singles ← A0.Child
A^2+ ← A^2+ − w.CNSet
FOR each neighbor x of w DO
  BEGIN
    IF x.CNSet.Child.CurrentVertex ≠ w THEN
      BEGIN
        x.CNSet.Child ← CreateNewEmptyCNSet
        x.CNSet.Child.CurrentVertex ← w
        END
    IF x.CNSet+ ≠ ∅ THEN
      BEGIN
        x.CNSet+.Child ← CreateNewEmptyCNSet
        x.CNSet+.Child.CurrentVertex ← w
          A^2+ ← A^2+ ∪ x.CNSet+
        x.CNSet+ ← x.CNSet+.Child
        END
    IF x.CNSet = A_0 THEN
      A_0 ← A_0 − {x}
    IF x ∈ W THEN
      A^2+ ← A^2+ ∪ x.CNSet
    x.CNSet.Child ← x.CNSet.Child ∪ {x}
    x.CNSet ← x.CNSet − {x}
    x.CNSet ← x.CNSet.Child
    END
```

Fig. 2.18

The algorithm (**B1**) to determine $M_W, A_W^1, A_W^{2+}, A_W^0$

Algorithm B1 computes $CNSet_{W \cup \{w\}}(x)$, $CNSet+_{W \cup \{w\}}(x)$, and $A^1_{W \cup \{w\}}(x)$ for each $x \in V$ in addition to $A^{2+}_{W \cup \{w\}}$ and $A^0_{W \cup \{w\}}$. This is done by maintaining for each vertex x the fields *x.CNSet*, *x.CNSet+* and *x.singles* in Algorithm B1 computes $CNSet_{W \cup \{w\}}(x)$, $CNSet+_{W \cup \{w\}}(x)$, and $A^1_{W \cup \{w\}}(x)$ for each $x \in V$ in addition to $A^{2+}_{W \cup \{w\}}$ and $A^0_{W \cup \{w\}}$. This is done by maintaining for each vertex x the fields *x.CNSet*, *x.CNSet+* and *x.singles* in addition to maintaining the variables A_0 (a *CNSet*) and A^{2+} (a set of *CNSets*). Finally, each *CNSet* has a *CurrentVertrex* field (to indicate the most recent vertex adjacent to any vertex in the *CNSet*) and a *child* field (pointing to another *CNSet*) for splitting *CNSet* s.

After execution of algorithm B1, for each vertex $x \in V - (W \cup \{w\})$ $x.singles = A^1_{W \cup \{w\}}(x)$ and $x.CNSet \cup x.CNSet+ = M_{W \cup \{w\}}(x)$. Also, after execution $A_0 =$

(1) $Aug \leftarrow \varnothing$
(2) IF $A_W^{2+} \neq \varnothing$ THEN $Aug \leftarrow \{a\} \subseteq A_W^{2+}$
(3) ELSE BEGIN
(4) $w \leftarrow$ some element of W such that $M_W(w) \cup A_W^1(w) \neq \varnothing$
(5) FOR each $v \in A_W^1(w)$ DO
(6) FOR each $x \in N_{V-W}(v)$ DO
(7) IF $x \in (M_W - M_W(w)) \cup (A_W^1 - A_W^1(w))$ THEN
(8) BEGIN
(9) $Aug \leftarrow \{v, x\}$
(10) break to line 37
(11) END
(12) FOR each $v \in M_W(w)$ DO
(13) FOR each $x \in N_{V-W}(v)$ DO
(14) IF $x \in (M_W - M_W(w)) \cup (A_W^1 - A_W^1(w))$ AND $(x, w) \notin E$ THEN
(15) BEGIN
(16) $Aug \leftarrow \{v, x\}$
(17) break to line 37
(18) END
(19) FOR each $y \in N_W(w)$ DO
(20) FOR each $x \in M_W(y)$ DO
(21) FOR each $v \in M_W(w)$ DO
(22) IF $(x, v) \notin E$ THEN
(23) BEGIN
(24) $Aug \leftarrow \{v, x\}$
(25) break to line 37
(26) END
(27) DO a BFS from $M_W(w) \cup A_W^1(w)$ through A_W^0 until an element
(28) a_k of $(M_W - M_W(w)) \cup (A_W^1 - A_W^1(w))$ is found
(29) IF such an a_k is found THEN
(30) BEGIN
(31) $Aug \leftarrow$ a primitive path from a_k to $M_W(w) \cup A_W^1(w)$
(32) break to line 37
(33) END
(34) END
(35) IF $Aug = \varnothing$ THEN
(36) Declare G is not a 4-prime graph and HALT
(37) IF $|Aug| = 1$ THEN Output (Rule 1, Lemma 2)
(38) ELSE IF $Aug \cap M_W = \varnothing$ THEN Output (Rule 3, Lemma 2)
(39) ELSE IF $|Aug \cap A_W^1| = 1$ THEN Output (Rule 4, Lemma 2)
(40) ELSE IF $|Aug| = 2$ THEN Output (Rule 2, Lemma 2)
(41) ELSE Output (Rule 5, Lemma 2)
(42) $W_1 \leftarrow W \cup Aug$

Fig. 2.19
The algorithm (**C1**) to determine how to augment W

$A^0_{W \cup \{w\}}$ and the union of all the vertices in all the subsets of A^{2+} forms $A^{2+}_{W \cup \{w\}}$. Furthermore, each field (*CNSet*, *CNSet*+, and *singles*) will have been updated for each vertex in V. The only line which is not O(1) is line 5 which introduces a multiplicative factor of deg(w). Thus, algorithm B1 is O(deg(w)) over one pass of the algorithm and hence O($|E|$) if taken once over all the vertices in the graph.

The second part of the process, algorithm C1, takes as input a graph $G = (V, E)$ and $W \subseteq V$ where G/W is 4–prime and $M_W(w)$ and $A^1_W(w)$ are known for each $w \in W$ along with A^{2+}_W and A^0_W. It produces as output one of the rules in Lemma 2 along with the set, *Aug*, by which W is to be augmented.

If G/W is 4-prime and known, algorithm **C1** will agument W by *Aug* according to one of the rules in Lemma 2 so that the new induced graph will also be 4–*prime*. Although Algorithm C1 produces a rule of Lemma 2 its organization reflects Lemma 1. Line 2 takes care of rule 1, Lemma 1 (rule 1, Lemma 2). Lines 5-11 account for part of rule 2ii, Lemma 1 ($k = 0$ of rule 4, Lemma 2) while $x \in A^1_W$ in lines 12-18 accounts for the other half of rule 2ii, Lemma 1 ($k = 0$ of rule 4, Lemma 2). $x \notin A^1_W$ in lines 12-18 accounts for part of rule 2i, Lemma 1 (rule 2, Lemma 2) while lines 19-26 account for the remaining part of rule 2i, Lemma 1 (rule 2, Lemma 2). Finally, lines 27-33 account for rule 2iii, Lemma 1 ($k \geq 1$ of rules 3,4, and 5, Lemma 2). Each section, save for line 19-26, is a straightforward application of BFS so that the running time for those sections is clearly O(E). Lines 19-26 are analogous to lines 10-16 in algorithm A (the first 4-decomposition algorithm). No edge is examined more than once, and the algorithm terminates immediately upon detection of a non-edge. Hence there are no more than $|E|+1$ checks. Thus, one pass over algorithm C1 takes O($|E|$) time.

Combining algorithms B1 and C1 yields another O($|V| \times |E|$) algorithm to determine if a graph is 4-prime. Furthermore, the combination (along with O($|E|$) preprocessing) yields a basic 4–constructing sequence $W_0, W_1, \cdots, W_k = V$ subject to the rules of Lemma 2. By modifying this sequence (O($|V|$) time) according to the Cor. 1, we produce a maximal 4–constructing sequence. This maximal 4–constructing sequence will be used in the following chapter to prove the existence of an O($|V| \times |E|$) algorithm to recognize circle graphs.

By modifying algorithms B1 and C1 appropriately (to algorithms B2 and C2 respectively), it is possible to obtain an O($|V|^2$) algorithm for determining whether a graph $G = (V, E)$ is 3–prime and for producing a maximal 3–*constructing* sequence if it is. Let z be a vertex not in V and form $G' = (V', E')$ where $V' = V \cup \{z\}$ and $E' = E \cup (V \times \{z\})$. G' is 4–prime iff G is 3–prime. If G contains an embedded P_4, then z together with the vertices of an embedded P_4 from an embedded $C_5^{2,3}$ in G'.

Each vertex $v \in V'$ in algorithm B2 (that is, the modified algorithm B1) will have an additional field associated with it: *v.untested*, which is the set of all vertices in $V - v.CNSet$ which have not yet been tested in the fashion to be described below. Thus, after the *untested* field of each vertex is initialized, the only changes occur when the *CNSet* of a given vertex, v, is split. In this case, each vertex appearing in the old *CNSet* of v but not v's *CNSet* after splitting is added to *v.untested*. Thus, the total amount of work in adding vertices to the *untested* field over all passes of algorithm C1 is O($|V| \times |V|$). If algorithm C1 were run on G', lines 27-33 need not be executed since $A_W^0 = \emptyset$ in G' when $z \in W$. By modifying the remaining parts of algorithm C1 to yield algorithm C2 we obtain an O($|V| \times |V|$) algorithm for recognizing 3–prime graphs.

```
( 1)  Aug ← ∅
( 2)  IF A_W^{2+} ≠ ∅ THEN Aug ← {a} ⊆ A_W^{2+}
( 3)     ELSE IF M_W ≠ ∅ THEN
( 4)        BEGIN
( 4)           w ← some element of W such that M_W(w) ≠ ∅
( 5)           FOR each v ∈ M_W(w) DO
( 6)              FOR each y ∈ v.untested DO
( 7)                 IF y ∈ A_W^1 − A_W^1(w) AND (v,y) ∈ E THEN
( 8)                    BEGIN
( 9)                       Aug ← {v,x}
(10)                       break to line 22
(11)                       END
(12)                    ELSE IF y ∈ M_W − M_W(w) AND
(13)                          exactly one of m,w is adjacent to y THEN
(14)                       BEGIN
(15)                          Aug ← {v,x}
(16)                          break to line 22
(17)                          END
(18)                       ELSE remove y from v.untested
(19)           END
(20)  IF Aug = ∅ THEN
(21)     Declare G is not a 4-prime graph and HALT
(22)  IF |Aug| = 1 THEN Output (Rule 1, Lemma 3)
(23)     ELSE IF Aug ∩ M_W = ∅ THEN Output (Rule 4, Lemma 3)
(24)        ELSE IF Aug ∩ M_W(z) = ∅ THEN Output (Rule 2, Lemma 3)
(25)           ELSE Output (Rule 3, Lemma 3)
(26)  W_1 ← W ∪ Aug
```

Fig. 2.20

The algorithm (**C2**) to determine how to augment W

Since each of the test takes $O(1)$ time and no pair of vertices is tested more than once (since no removed vertex is re-added to the *untested* field of the vertex it was removed from), over all passes algorithm C2 takes $O(|V| \times |V|)$ time. The modification of the basic 3–constructing sequence to a maximal 3–constructing sequence takes $O(|V|)$ time using the method outlined in the proof of Cor. 2. Hence, using algorithms B2 and C2 (along with the $O(|V| \times |V|)$ initializations of the untested fields and the $O(|E|)$ initialization in finding a P_4) we have an $O(|V| \times |V|)$ algorithm for recognizing 3–prime graphs and producing a maximal 3–constructing sequence for them.

3. Circle Graphs

3.1. Basic Ideas

Let V be a set of size n. Define a V-*2seq* to be any sequence of length $2n$ with each element of V appearing exactly twice. Suppose S is a V-2seq and $W \subseteq V$. The notation $S-W$ is used to designate the $(V-W)-2seq$ which is a subsequence of S. We define two sets of operators on V-2seqs as follows: let $S=(s_0, s_1, s_2, \cdots, s_{2n-1})$ be a V-2seq. Then $R_k(S) = (s_{[k]}, s_{[k+1]}, s_{[k+2]}, \cdots, s_{[k+(2n-1)]})$ and $F_k(S) = (s_{[k]}, s_{[k-1]}, s_{[k-2]}, \cdots, s_{[k-(2n-1)]})$ where $[i] \equiv i_{mod\, 2n}$. Thus, there are $4n$ distinct operators. These operators are related as follows:

$$R_k(R_l(S))=R_{k+l}(S)$$

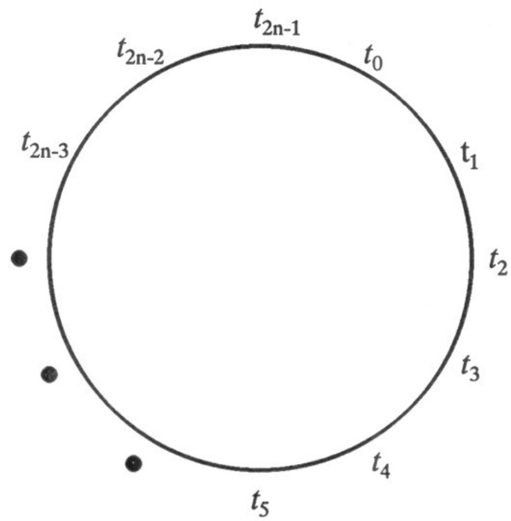
$$F_k(F_l(S))=R_{l-k}(S)$$

$$F_k(R_l(S))=F_{l+k}(S)$$

$$R_k(F_l(S))=F_{l-k}(S)$$

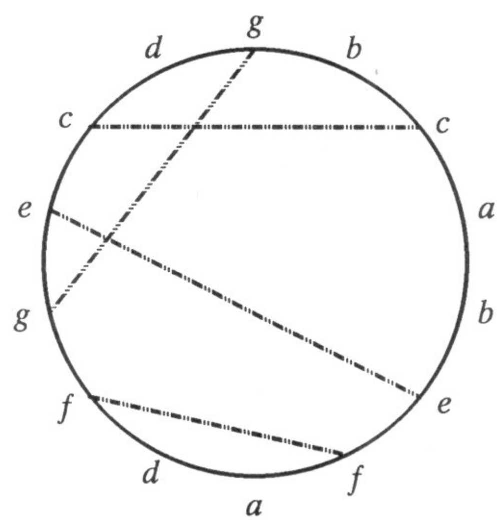
One may note that $\{R_k{:}0 \le k<2n\} \cup \{F_k{:}0 \le k<2n\}$ forms a group of order $4n$ isomorphic to D_{2n}, the dihedral group of size $4n$.

Suppose S is a V-2seq. Define $[S] = \{T{:}T = R_k(S)$ or $T = F_k(S)$ for some $k \in \{0, 1, \cdots, 2|V|-1\}\}$. By utilizing the relationships above, it is easy to see that an equivalence class has been defined. $[S]$ may be thought of as a circular sequence. It is convenient to depict $[S]$ as shown in Fig. 3.1 where $(t_0, t_1, t_2, \cdots, t_{2n-1}) \sim S$. Thus, S may be found on the circle if read off in the appropriate direction (clockwise, counterclockwise) from the appropriate starting element. Any depiction of $[S]$ as in Fig. 3.1 is called a circular model for $[S]$. To distinguish between adjacency in graphs and $V-2seqs$, the word *adjacent* is reserved for graphs and *next-to* is reserved for $V-2seqs$ and the corresponding equivalence classes. Similar to the concept of a local neighborhood, $N_W(U)$ (where $U, W \subseteq V$), is the $A(ssociation)I(nterval)([S], U)$.



A way of depicting $[(t_1, t_2, t_3, \ldots, t_n)]$

Fig. 3.1



[S]

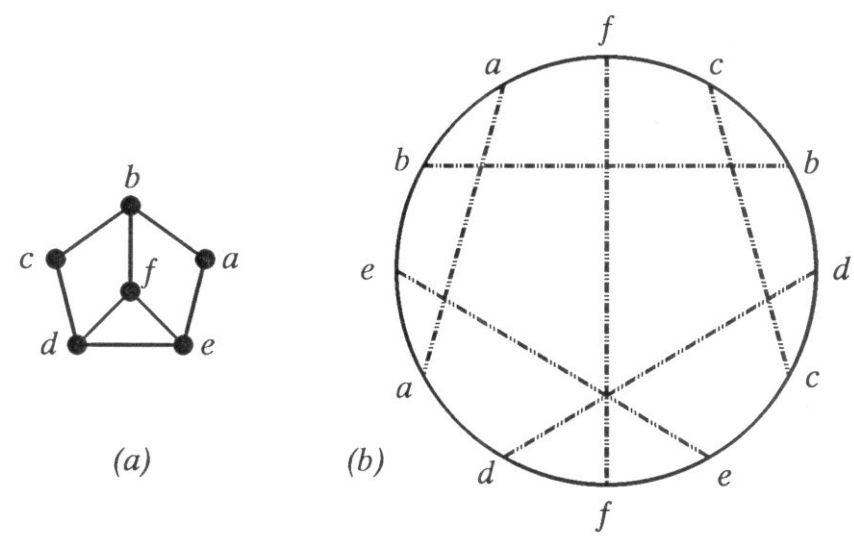
$AI([S], \{f\}) = \{\{a, e\}, \{d, g\}\}$
$AI([S], \{f, g\}) = \{\{a, e\}, \{d, e\}, \{b, d\}\}$
$AI([S], \{f, c\}) = \{\{a, e\}, \{d, g\}, \{d, e\}, \{a, b\}\}$
$AI([S], \{e, f, g\}) = \{\{a, b\}, \{c, d\}, \{b, d\}\}$

Examples of Association Intervals

Fig. 3.2

$AI([S], U) = \{\{v_1, v_2\} : v_1, v_2 \in V - U$ and there exists $w \in U$ such that v_1 and v_2 are next-to the same occurrence of w in $[S - (U - \{w\})]\}$ (see Fig. 3.2 for examples). In English, this means the set of all pairs of elements next-to each other in $[S - U]$ that *had* at least one element of U between them. Thus, if $x \in V$ and the two occurrences of x in $[S]$ are not next-to each other, then $|AI([S], \{x\})| = 2$.

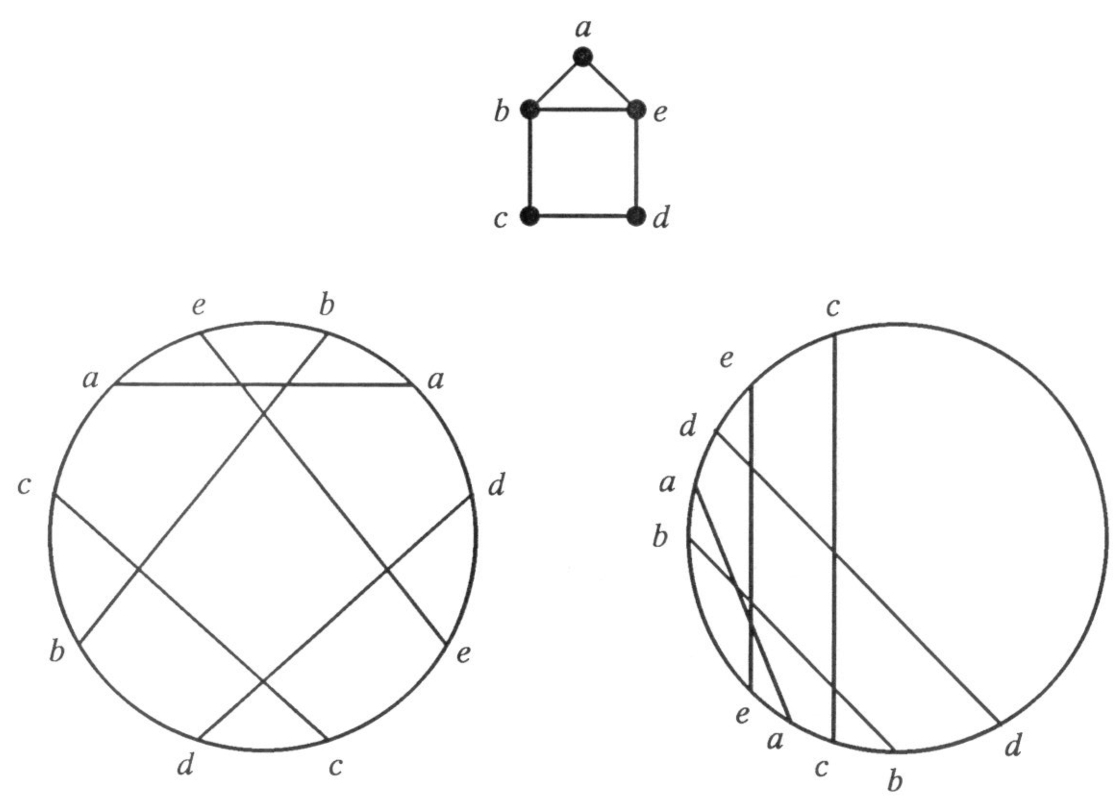
For each V-2seq, S, define a graph $G(S)$ as follows: the vertex set of $G(S)$ is V and two vertices v and w in V are adjacent iff $vwvw$ or $wvwv$ is a subsequence of S. If $S \sim T$, then $G(S) = G(T)$. Thus, we define $G([S]) = G(S)$. The type of function that G is (i.e. is its argument a V-2seq or an equivalence class?) will be apparent from context. $G = (V, E)$ is a *circle graph* iff $G = G(S)$ for some V-2seq, S. In this event we say that [S] is a model for G (see Fig. 3.3 for an example). We say that G is *uniquely representable* if $T \sim S$ whenever $G(T) = G(S)$. What it means for a circle graph to be uniquely representable is that there is essentially only one model for it.



A circle graph along with a model for it

Fig. 3.3

The reason for the name circle graph becomes clear upon examination of Fig. 3.3(b). For each pair of identical elements in the circular sequence, draw a chord between them. The condition of an alternating v and w subsequence is the same as the condition that the corresponding chords intersect. Given a circle with chords, a circle graph is formed by taking the chords as the vertex set the set of the graph, with edges being adjacent iff the corresponding chords intersect. A graph is a circle graph iff it is isomorphic to any graph formed in this fashion. Thus, there is a completely geometric interpretation of circle graphs as intersection graphs. The operator R_k corresponds to a physical rotation of the circle while F_k corresponds to flipping the circle (an easy way to picture this is to consider cutting the circle out of the paper, picking it up, turning it over, and placing it back down (with the appropriate rotation)). Again, it is clear that rotations and flips of the circle do not affect the circle graph generated from the circle. Furthermore, the spacing between the endpoints of the chords does not affect the circle graph generated since the spacing does not affect the property of intersection of the chords (see Fig. 3.4 for an example).

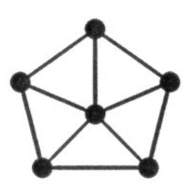


A circle graph along with two equivalent models for it

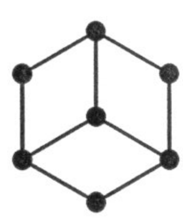
Fig. 3.4

A question which immediately arises is, are all graphs circle graphs? The answer is no, and a smallest example, shown below (Fig. 3.5), has six vertices. The fact that a non-circle graph must have at least 6 vertices follows from Fact 17 (next page): the smallest non-circle graph graph must be 4-prime by Fact 17, and there are only three 4-prime graphs of size less than 6 (They are shown in Fig. 1.6), all of which are circle graphs. The Central Lemma (1) allows one to quickly generate all the 4-prime graphs of size 6. However, by simple counting arguments one may see that there is exactly one way to add a vertex to each 4-prime graph (of size 5) to obtain a 4-prime non-circle graph of size 6. The example shown in Fig. 3.5 was obtained from a C_5 (or *tepee*). The only only other non-circle graph of size 6 may be determined from a *house* (it is the complement of a C_6).

Another central question, which is not answered so easily, is when does a circle graph have only one model? For example, $P_4(a, b, c, d)$ is uniquely representable while $P_5(a, b, c, d, e)$ is not, but C_n is uniquely representable for all n (Fig. 3.6). In fact, each graph in Fig. 1.5 and in Fig. 1.6 is uniquely representable. The answer to this question is intertwined with graph decomposition, and it forms one of the two central issues of this chapter. The other is a polynomial-time algorithm to recognize and construct a/the model of a circle graph. There are several easy to see, but nevertheless important, facts which will be utilized in the proof of the central theorem (5) on circle graphs. These follow below.



(a)

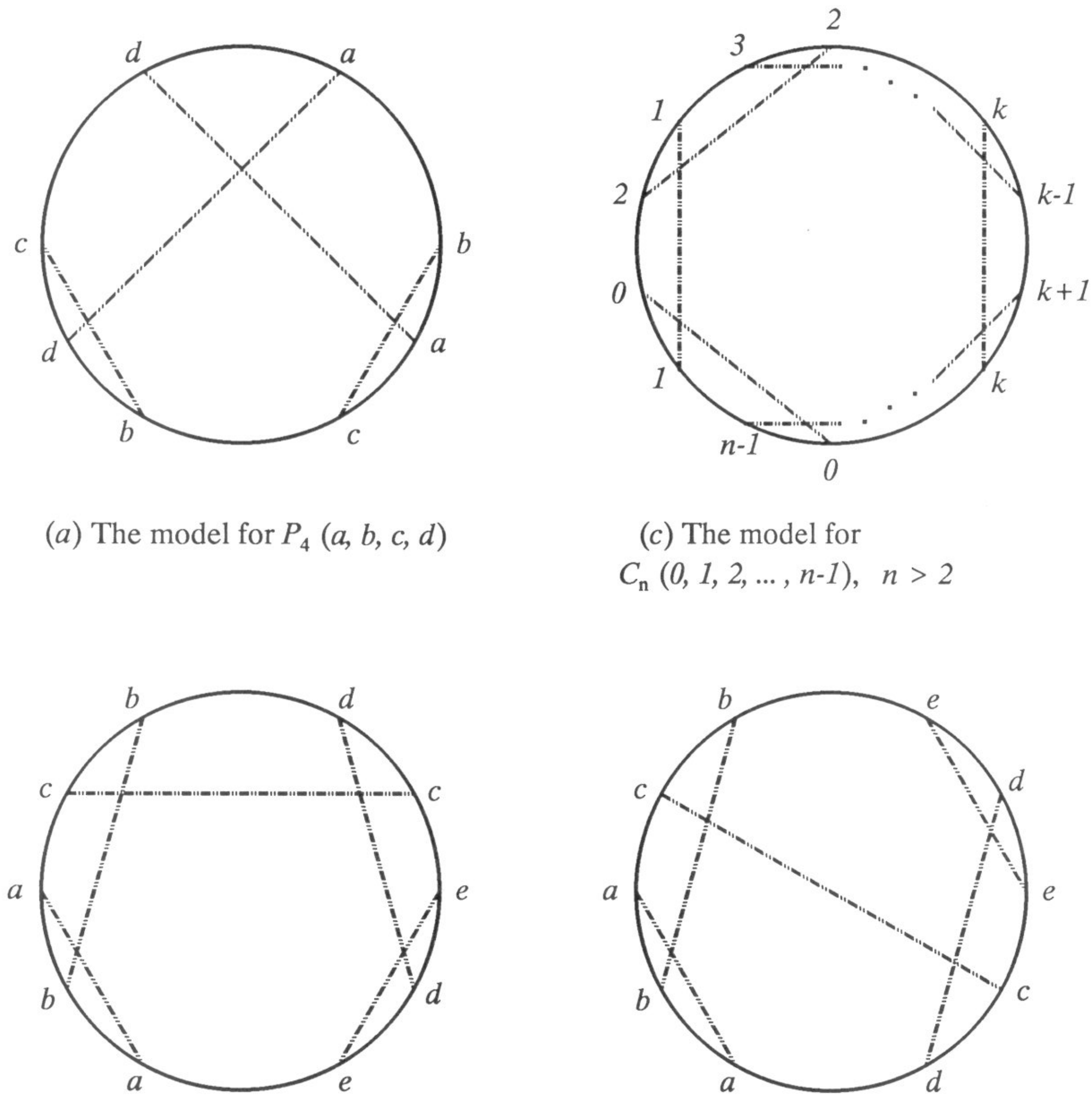


(b)

(a) A non-circle graph of size 6
(b) A non-circle graph of size 7
Fig. 3.5

Fact 17. If G/V 4–factors into G/V_1 and G/V_2 then G is a circle graph iff G/V_1 and G/V_2 are each circle graphs.

Proof: The only if follows immediately since G/V_1 and G/V_2 are each induced subgraphs of G/V. To prove the other direction, suppose that the model for G/V_1 is $[(v_2, w_1^1, w_2^1, \cdots, w_{i_1}^1, v_2, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ and the model for G/V_2 is $[(v_1, w_1^2, w_2^2, \cdots, w_{i_2}^2, v_1, w_1^0,$

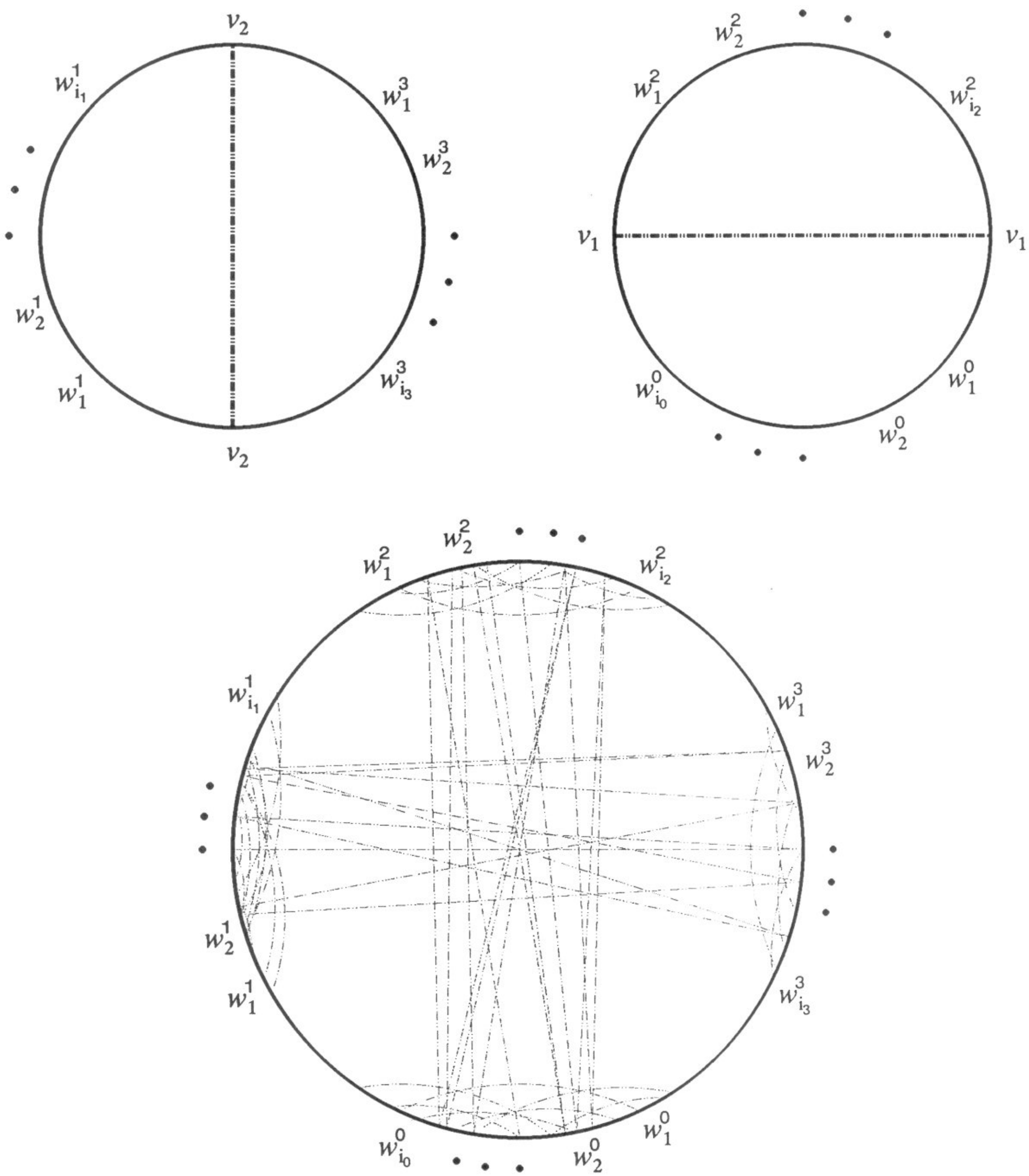


(a) The model for P_4 (a, b, c, d)

(c) The model for C_n ($0, 1, 2, \ldots, n$-1), $n > 2$

(b) The two non-equivalent models for P_5(a, b, c, d, e)

Fig. 3.6



Proof of Fact 17

Fig. 3.7

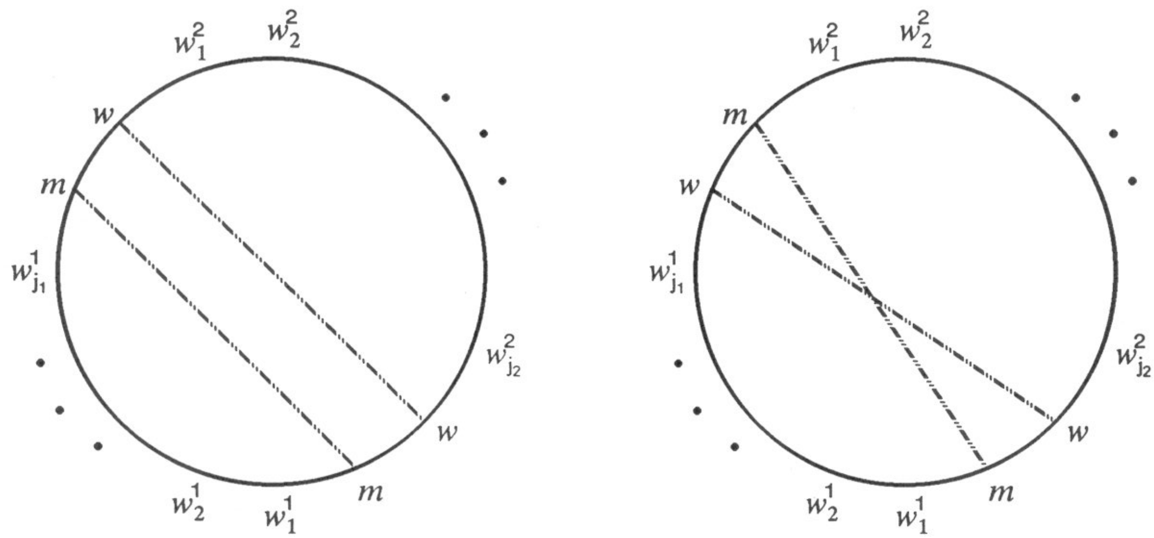
$w^0_2, w^0_{i_0})]$ where $v_2 \in V_2$ and $v_1 \in V_1$. Then we may combine the two models in the following fashion to produce a model for G/V: $[(w^1_1, w^1_2, \cdots, w^1_{i_1}, w^2_1, w^2_2, \cdots, w^2_{i_2}, w^3_1, w^3_2, \cdots, w^3_{i_3}, w^0_1, w^0_2, \cdots, w^0_{i_0})]$ (Fig. 3.7). Viewed another way, this Fact shows us a way to decompose circle graphs in a pictoral fashion. It also demonstrates that 4-decomposeable graphs (of size 5 or larger) are not uniquely representable. Thus, we may restrict our attention to 4-prime graphs.

Fact 18. If G/W is a 4-prime circle graph and $m \in M_W(w)$ then $G/(W \cup \{m\})$ is a circle graph and in each equivalence class $[S']$ of $(W \cup \{m\})$−2seqs such that $G([S']) = G/(W \cup \{m\})$, each of the two occurences of m is next-to a distinct occurrence of w.

Proof of Fact 18. Let $[S]$ be a model for G where $S = (w, w_1^1, w_2^1, \cdots, w_{j_1}^1, w, w_1^2, w_2^2, w_{j_2}^2)$. Then $S' = (w, m, w_1^1, w_2^1, \cdots, w_{j_1}^1, x, y, w_1^2, w_2^2, w_{j_2}^2)$ (where $\{x, y\} = \{m, w\}$ and $x = w$ iff $(w, m) \in E$) is a model for $G/(W \cup \{m\})$ (Fig. 3.8). To prove that each occurrence of m is next-to a dintinct occurrence of w assume that $S' = (w, w_1^1, w_2^1, \cdots, w_{i_1}^1, x, w_1^2, w_2^2, \cdots, w_{i_2}^2, y, w_1^3, w_2^3, \cdots, w_{i_3}^3, m, w_1^0, w_2^0, \cdots, w_{i_0}^0)$ (Fig. 3.9), $\{x, y\} = \{m, w\}$, $G(S') = G/(W \cup \{m\})$, and $S' - \{m\} = S$. Form $W^k = \{w_j^k : j \in \{1, \cdots, i_k\}\}$. Since m and w are W-similar, $W^i \cap W^{[i+1]} = \varnothing$ for $i \in \{0, 1, 2, 3\}$ (where $[i+1] \equiv i+1_{mod\,4}$). Also, each vertex in $W^1 \cap W^3$ is adjacent to each vertex in $W^2 \cap W^0$ so $(W^1 \cup W^3 - (W^1 \cap W^3),\ W^1 \cap W^3,\ (W^2 \cap W^0) \cup \{w\},\ W^2 \cup W^0 - (W^2 \cap W^0))$ is a 4-decomposing partition.

If $W^1 \cup W^3 = \varnothing$, then (since G is connected) each occurrence of w is next-to a distinct occurrence of m in S'. If $W^1 \cup W^3 = \{z\}$ then $W^1 \cap W^3 = \{z\}$ since G is connected, but then we would have z either an articulation point in G/W or W-similar to w, a contradiction. Finally, if $|W^1 \cup W^3| > 1$, then $W^1 \cup W^3 = W - \{w\}$ and $W^2 \cup W^0 = \varnothing$ since G/W is 4-prime, and hence we see that each occurrence of m is next-to an occurrence of w.

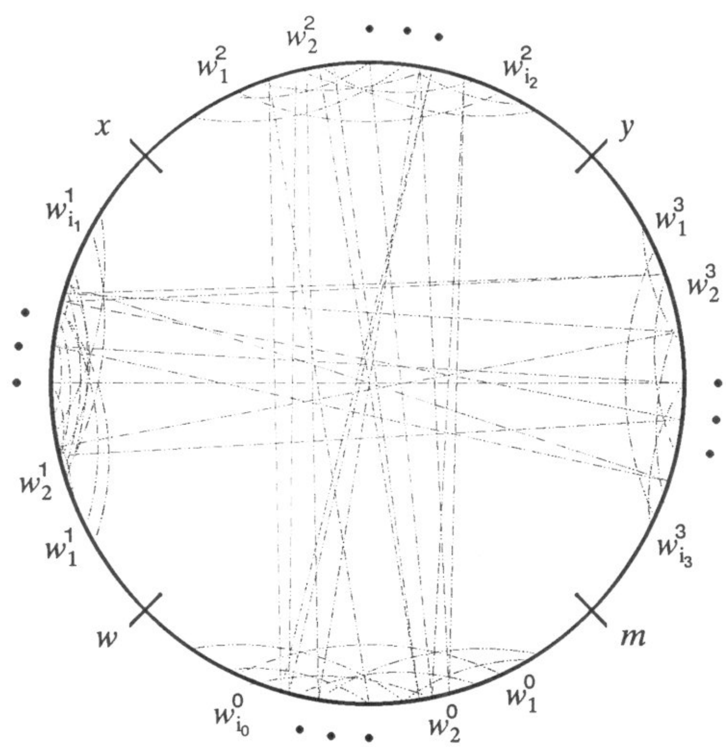
Fact 19. Suppose G/W is a 4-prime circle graph, and $V - W = \{a\} = A_W^1(w)$ for some $w \in W \subseteq V$. If $[S]$ is an equivalence class such that $G([S]) = G/W$, then there are at most two distinct equivalence classes $[S'_1]$ and $[S'_2]$ such that $[S] = [S'_1 - \{a\}] = [S'_2 - \{a\}]$ and $G([S'_1]) = G([S'_2]) = G$.



(*a*) (*b*)

The two possible models for $G / (W \cup \{m\})$ in the proof of Fact 17, depending on whether or not $(w, m) \in E$.

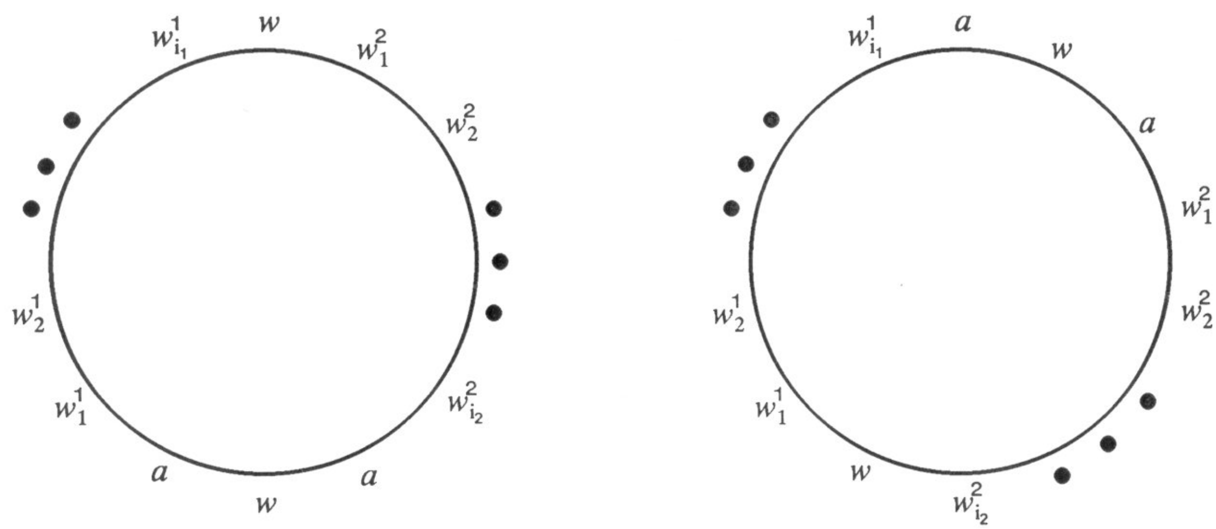
Fig. 3.8



$\{x, y\} = \{m, w\}$ $x = w$ iff $(w, m) \in E$

Fig. 3.9

Proof. There can be no more that two distinct equivalence classes $[S'_1]$ and $[S'_2]$ since G/W contains no articulation points. If $[S] = [(w, w_1^1, w_1^2, \cdots, w_{i_1}^1, w, w_2^1, w_2^2, \cdots, w_{i_2}^2)]$ then $[S'_1] = [(a, w, a, w_1^1, w_1^2, \cdots, w_{i_1}^1, w, w_2^1, w_2^2, \cdots, w_{i_2}^2)]$ and $[S'_2] = [(w, w_1^1, w_1^2, \cdots, w_{i_1}^1, a, w, a, w_2^1, w_2^2, \cdots, w_{i_2}^2)]$ (Fig. 3.10).



The two ways of adding $a \in A_W^1(w)$

Fig. 3.10

Fact 20. If $G = (V, E)$ is a 4-prime circle graph and S is a V-2seq such that $G(S) = G$ and $v, w \in V$ such that v is next-to w in $[S]$, then there is a unique $T \in [S]$ such that $T = (v, w, t_1, t_2, \cdots, t_{2|V|-1})$.

Proof: Both occurrences of w in $[S]$ are not next-to the same occurrence of v since w is adjacent to at least two vertices. Similarly, both occurrences of v in $[S]$ are not next-to the same occurrence of w since v is adjacent to at least two vertices. Finally, since v is not similar to w, each occurrence of v is not next-to a distinct occurrence of w. Thus, there is only one location within $[S]$ where v is next-to w; hence, it uniquely specifies the remainder of the sequence.

The theorem which is central to circle graphs (and lends itself to an $O(|V| \times |E|)$ recognition algorithm) is the following:

Theorem 5: Circle graphs with five or more vertices are uniquely representable iff they are 4-prime.

We have shown the only if part in proving Fact 17. We shall prove the if part of this theorem by induction on the size of all circle graphs. It may be easily verified that each graph in Fig. 1.6 is uniquely representable. The induction hypothesis will be: all 4-prime circle graphs with i vertices are uniquely representable. For the basis step, $i = 5$, there are exactly three 4-prime graphs and all are circle graphs (they are shown in Fig. 1.6). We use the same approach as for the previous theorems (3 and 4). We assume the truth of the induction hypothesis for all $i \le j$ and show this implies the truth of the induction hypothesis for $i = j+1$. Thus, suppose $G = (V, E)$ is a prime circle graph where $|V| = j+1$ and $(V_0, V_1, \cdots, V_k)$, $k \ge 0$, is a maximal constructing sequence subject to Cor 1. If $k = 0$ then G/V_k is isomorphic to one of the graphs in Fig. 1.6, which we have verified above as uniquely representable.

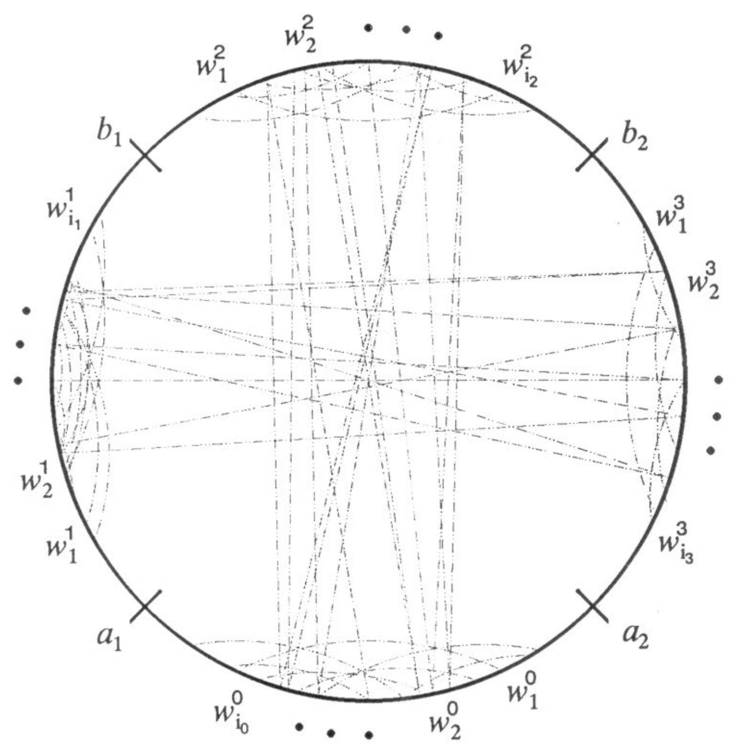
Thus, suppose $k \ge 1$. By the induction hypothesis, G/V_{k-1} is uniquely representable. We need only verify that application of rules 1, 2, or 3 in Lemma 2 (by Cor. 1) still yield a uniquely representable circle graph. Recall that $V = V_k$, and for the remainder of this proof we denote V_{k-1} by W.

Case 1: $V - W = \{a\}$

Suppose that there are two distinct ways to place $a \in A_W^{2+}$. Form the $W \cup \{a_1, a_2\}$-2seq $S'' = (a_1, w_1^1, w_2^1, \cdots, w_{i_1}^1, b_1, w_1^2, w_2^2, \cdots, w_{i_2}^2, \cdots, b_2, w_1^3, w_2^3, \cdots, w_{i_3}^3, a_2, w_1^0, w_2^0, \cdots, w_{i_0}^0)$ where $\{b_1, b_2\} = \{a_1, a_2\}$ with the following characteristics: $G(S - \{a_1, a_2\}) = G/W$, $G(S - \{a_2\})$ is isomorphic to G/V with a_1 in one of the two distinct positions a could occupy in a model for G (where the only difference is that a is now a_1), and

$G(S-\{a_1\})$ is isomorphic to G/V with a_2 in the other distinct position a could occupy in the model for G (with the only difference that a is now a_2). Form $W^k = \{w_j^k : j \in \{1, \cdots, i_k\}\}$ (see Fig. 3.11).

Clearly, each vertex in $W^1 \cap W^3$ is adjacent to each vertex in $W^2 \cap W^0$. Since a_1 and a_2



$\{b_1, b_2\} = \{a_1, a_2\}$

Fig. 3.11 Case 1 in the proof of Theorem 5

are W*-similar* (in $G(S'')$), $W^i \cap W^{[i+1]} = \varnothing$ for $i \in \{0, 1, 2, 3\}$ (where $[i+1] \equiv i+1_{mod\,4}$). Thus, $(W^1 \cup W^3 - (W^1 \cap W^3),\ W^1 \cap W^3,\ W^2 \cap W^0,\ W^2 \cup W^0 - (W^2 \cap W^0))$ is a 4-decomposing partition of W. Since (by assumption) there are two distinct ways to place a_1 and a_2, $W^1 \cup W^3 \neq \varnothing$ and $W^2 \cup W^0 \neq \varnothing$. Since G is connected it must also be the case that $W^1 \cap W^3 \neq \varnothing$ and $W^2 \cap W^0 \neq \varnothing$. Hence, $W^i \neq \varnothing$ for $i \in \{0,1,2,3\}$. Since a is an element of neither M_W nor of A_W^1, $|W^1 \cup W^3| \geq 2$ which means that the partition 4-factors G/W, a contradiction.

Case 2: Rule 2 is applied: $V-W = \{m_1, m_2\}$, $m_1 \in M_W(w_1)$, and $m_2 \in M_W(w_2)$.

Suppose S'' is a $(W \cup \{m_1, m_2\})$-2seq such that $G(S'') = G$. Then, by Fact 18, $S'' - \{m_2\}$ is a $(V \cup \{m_1\})$-2seq where each occurrence of m_1 is next-to a distinct occurrence of w_1. Similarly, $S'' - \{m_1\}$ is a $(V \cup \{m_2\})$-2seq where each occurrence of m_2 is next-to a distinct occurrence of w_2.

Case 2.1: $(w_1, w_2) \notin E$ (Fig. 3.12a).

Then $S'' = (x_1, y_1, w_1^1, w_2^1, \cdots, w_{i_1}^1, x_2, y_2, w_1^2, w_2^2, \cdots, w_{i_2}^2, x_3, y_3, w_1^3, w_2^3, \cdots, w_{i_3}^3, x_0, y_0, w_1^0, w_2^0, \cdots, w_{i_0}^0)$ where $\{x_1, y_1\} = \{m_1, w_1\}$, $\{x_0, y_0\} = \{m_2, w_2\}$, $w_1 \in \{x_2, y_2\}$, and $w_2 \in \{x_3, y_3\}$. Form $W^k = \{w_j^k : j \in \{1, \cdots, i_k\}\}$. The reason for this phrasing is that without loss of generality we may assume $|W^1|$, $|W^3|$, and $|W^0|$ are all greater than 0. There is nothing to preclude one occurrence of w_1 from being next-to an occurrence of w_2 in $S'' - \{m_1, m_2\}$. There are two cases:

Case 2.1.1: $W^2 \neq \emptyset$

In this case $\{x_2, y_2\} = \{m_1, w_1\}$ and $\{x_3, y_3\} = \{m_2, w_2\}$. But this means that vertices m_1 and m_2 are not adjacent in $G(S'')$. That is, $G(S'') \neq G$, which is a contradiction.

Case 2.1.2: $W^2 = \emptyset$

Since m_1 is W-similar to w_1 and adjacent to m_2, $m_1 \in \{x_3, y_2\}$. Similarly, since m_2 is W-similar to w_2 and adjacent to m_1, $m_2 \in \{x_3, y_2\}$. The only solution of these consistent with $w_1 \in \{x_2, y_2\}$ and $w_2 \in \{x_3, y_3\}$ is $w_1 = x_2$, $m_2 = y_2$, $m_1 = x_3$, and $w_2 = y_3$. With these substitutions, $G(S'') = G$.

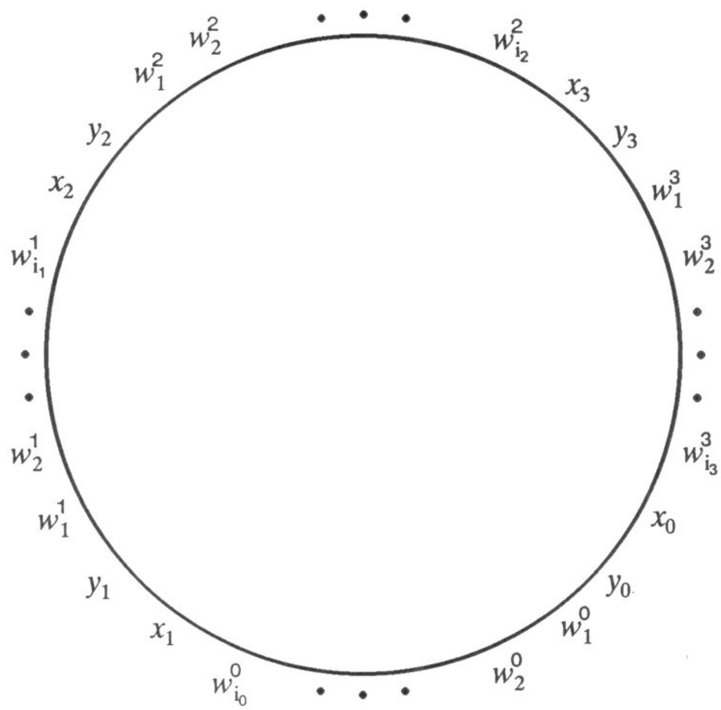
Case 2.2: $(w_1, w_2) \in E$ (this case is almost identical to the previous one) (Fig. 3.12b).

$w_1 \in \{x_2, y_2\}$

$w_2 \in \{x_3, y_3\}$

$\{x_2, x_3, y_2, y_3\} =$
$\{m_1, m_2, w_1, w_2\}$

a) $(w_1, w_2) \notin E$



$\{m_1, w_1\} = \{x_1, y_1\}$

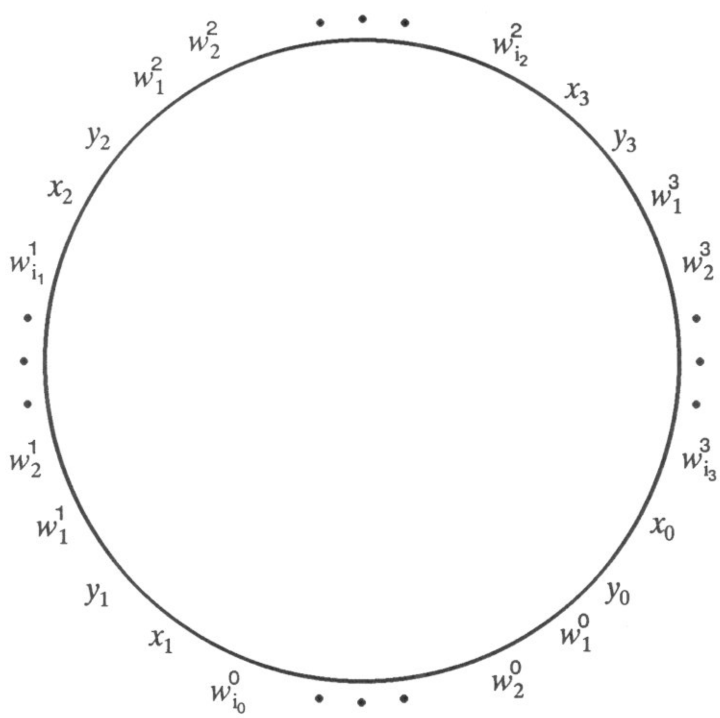
$\{m_2, w_2\} = \{x_0, y_0\}$

$w_2 \in \{x_2, y_2\}$

$w_1 \in \{x_3, y_3\}$

$\{x_2, x_3, y_2, y_3\} =$
$\{m_1, m_2, w_1, w_2\}$

b) $(w_1, w_2) \in E$



$\{m_1, w_1\} = \{x_1, y_1\}$

$\{m_2, w_2\} = \{x_0, y_0\}$

Case 2.1 and 2.2 of Theorem 5

Fig. 3.12

Then $S'' = (x_1, y_1, w_1^1, w_2^1, \cdots, w_{i_1}^1, x_2, y_2, w_1^2, w_2^2, \cdots, w_{i_2}^2, x_3, y_3, w_1^3, w_2^3, \cdots, w_{i_3}^3, x_0, y_0, w_1^0, w_2^0, \cdots, w_{i_0}^0)$ where $\{x_1, y_1\} = \{m_1, w_1\}$, $\{x_0, y_0\} = \{m_2, w_2\}$, $w_1 \in \{x_3, y_3\}$, and $w_2 \in \{x_2, y_2\}$. Form $W^k = \{w_j^k : j \in \{1, \cdots, i_k\}\}$. The reason for this phrasing is that without loss of generality we may assume $|W^1|$, $|W^3|$, and $|W^0|$ are all greater than 0. There is nothing to preclude one occurrence of w_1 from being next-to an occurrence of w_2 in $S'' - \{m_1, m_2\}$. There are two cases:

Case 2.2.1: $W^2 \neq \emptyset$

In this case $\{x_3, y_3\} = \{m_1, w_1\}$ and $\{x_2, y_2\} = \{m_2, w_2\}$. But this means that vertices m_1 and m_2 are adjacent in $G(S'')$. That is, $G(S'') \neq G$, which is a contradiction.

Case 2.2.2: $W^2 = \emptyset$

Since m_1 is W-similar to w_1 and not adjacent to m_2, $m_1 \in \{x_3, y_2\}$. Similarly, since m_2 is W-similar to w_2 and not adjacent to m_1, $m_2 \in \{x_3, y_2\}$. The only solution of these consistent with $w_1 \in \{x_3, y_3\}$ and $w_2 \in \{x_2, y_2\}$ is $w_2 = x_2$, $m_1 = y_2$, $m_2 = x_3$, and $w_1 = y_3$. With these substitutions, $G(S'') = G$.

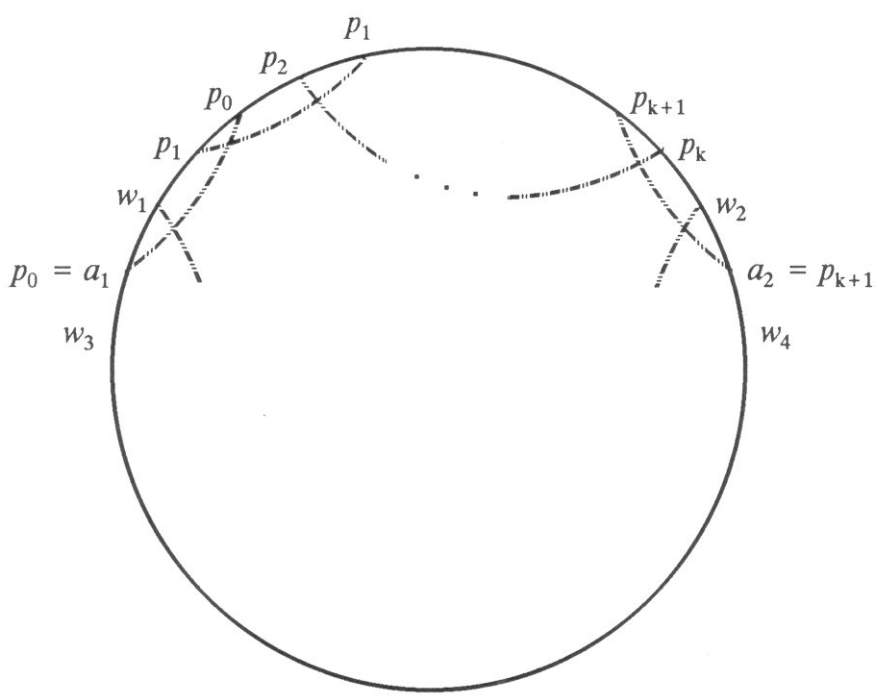
Thus, not only have we shown case 2, but we have also given a necessary and sufficient condition for when a prime circle graph may be successfully augmented using rule 2 of Lemma 2.

Case 3: Rule 3 is applied.

In this case let $A = \{p_1, p_2, \cdots, p_k\}$, $A' = A \cup \{a_1, a_2\}$, S' be a V-2seq such that $G(S') = G$ and $G(S' - A') = G/W$. Then the two occurrences of p_i are next-to each other in $[S' - (A' - \{p_i\})]$ for each $i \in \{1,2, \cdots, k\}$ since G/W is connected. Let $[T_i] = [S' - (A' - \{p_i\})]$ $(i \in \{1,2, \cdots, k\})$ so that $T_i = [(t_1^i, p_i, p_i, t_2^i, t_3^i, \cdots, t^i_{2|W|})]$ (where $W = \{t_j^i : j \in \{1, \cdots, 2|W|\}\}$). Then either $[(t_1^i, p_i, p_{i+1}, p_i, p_{i+1}, t_2^i, \cdots t^i_{2|W|})]$ or $[(t_1^i, p_{i+1}, p_i, p_{i+1}, p_i, t_2^i, \cdots t^i_{2|W|})]$ is equal to $[S' - (A' - \{p_i, p_{i+1}\})]$ for $i \in \{1,2, \cdots, k-1\}$.

Thus, $t^i{}_l = t^j{}_l$ for $i, j \in \{1,2, \cdots ,k\}$ and $l \in \{1, \cdots 2|W|\}$. By using Fact 19 along with the fact that p_1 is adjacent to a_1 and the fact that p_n is adjacent to a_2 we have that a_1 is adjacent to either t_1^1 or t_2^1 and a_2 is adjacent to the element of $\{t_1^1, t_2^1\}$ that a_1 is not adjacent to. Thus, $[(c_1, t_1, d_1, d_2, t_2, c_2, t_3, \cdots, t_{2|W|})] = [S' - A]$ where $\{c_1, c_2\} = \{d_1, d_2\} = \{a_1, a_2\}$ and $c_1 = a_1$ iff $(a_1, w_1) \in E$ and $d_1 = a_1$ iff $((a_1, w_1) \in E$ and $(w_1, w_2) \notin E)$ or $((a_1, w_2) \in E$ and $(w_1, w_2) \in E)$. If $k = 0$ we are done. Thus, suppose $k \geq 1$. In this event, let D equal the vertices of the shortest path from w_1 to w_2 (in G/W) together with A'. Then G/D is an n*−cycle* ($n \geq 5$) which, by the induction hypothesis, is uniquely representable. Thus, since $G/D = G(S' - (V - D))$ is prime, this fixes the order of the elements between w_1 and w_2 (by Fact 20). Hence, the order of elements in S' is also fixed so that $[S']$ is unique. I.e. G is uniquely representable (Fig. 3.13).

QED Theorem 5



How the chain is added

Fig. 3.13

3.2. Recognition Algorithm

Cor. 1 and Theorem 5 (together with Algorithms **B1** and **C1**) allow us to construct an O(|V||E|) algorithm for finding the model of a 4-prime circle graph (or determining that it isn't a circle graph). To find the model for a 4-prime circle graph, we first find a constructing sec-quence $(V_0, V_1, \cdots, V_k)$ subject to Cor. 1 (in O(|V||E|) time) and incrementally augment the model starting with the model for G/V_0. If rule 2 (of Cor. 1) is used, then case 2 in the proof of Theorem 5 gave an O(1) method for refining the model. If rule 3 is used, then case 3 in the proof of Theorem 5 gave an O(k) method for refining the model. Both of these two cases involve locating the endpoints of w_1 and w_2 next-to each other.

Thus, if there were an easy way to determine how to augment the model when rule 1 (of Cor. 1) was used, then an algorithm for the recognition of circle graphs would be demonstrated. Unfortunately, it is not as easy as it was for rules 2 and 3. The remainder of this chapter is devoted to demonstrating that when a vertex x is added according to rule 1, then it is possible to quickly determine how to augment the model for it. The idea behind it is that for each vertex in $A^{2+}{}_{V_i}$, we associate its location in the current model *if it were to be added*. In most cases, the addition of other vertices will cause no change in this information. One of the things that will be demonstrated is that if change is caused, however, the change can be updated quickly (2 - in the following paragraph). Therefore, the other thing that must be shown is that when a vertex first becomes a member of an $A^{2+}{}_{V_i}$ it is easy to determine its location in the model (1 - in the following paragraph).

Thus, let's examine the adding of a vertex, $x \in A^{2+}{}_{V_i}$ to a 4-prime circle graph G/V_i according to rule 1 (of Cor. 1). First, find the largest $l \le i$ such that the vertex to be added $x \notin A^{2+}{}_{V_l}$. If no such l exists, we note that $a \in A^{2+}{}_{V_0}$, and it takes O(1) time to produce a refinement of the model for $G/(V_0 \cup \{x\})$. What we will show is that:

(1) if $l \geq 0$, then it is possible to compute the refinement for the model for $G/(V_l \cup \{x\})$ from the model for G/V_l in O(1) time and

(2) if $l \leq h \leq i$ and the model for $G/(V_h \cup \{x\})$ and G/V_{h+1} are known, then it takes O(1) time to compute a refinement for the model for $G/(V_{h+1} \cup \{x\})$. To this end, we introduce three helping facts:

Fact 21: Given G/W is a 4-prime circle graph with a known model, $x \in A_W^{2+}$, $N_W(x) = \{w_1, w_2\}$, and w_1 and w_2 have occurrences next-to each other in the model for G/W. Then it takes O(1) time to compute the refinement of the model for $G/(W \cup \{x\})$.

Proof of Fact 21: Since $[(w_1, w_2, v_1, \cdots, v_{2|W|-2})]$ is the model for G/W, $[(x, w_1, w_2, x, v_1, \cdots, v_{2|W|-2})]$ is the model for $G/(W \cup \{x\})$.

Fact 22: Given G/W is a 4-prime circle graph with a known model, w_1 and w_2 have occurrences next-to each other in the model for G/W, and $x \in M_{W-\{w_2\}}(w_1) \cap A_W^{2+}$. Then it takes O(1) time to compute the refinement of the model for $G/(W \cup \{x\})$.

Proof of Fact 22: Suppose that the model for G/W is $[(w_1, w_2, w_1^1, w_2^1, \cdots, w_{i_1}^1, w_1, w_1^2, w_2^2, \cdots, w_{i_2}^2, w_2, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ where $i_1+i_2+i_3 = 2|W|-4$ and $i_1 i_2 i_3 > 0$ (Fig. 3.14a). If $(w_1, x) \in E$ then $[(w_1, w_2, x, w_1^1, w_2^1, \cdots, w_{i_1}^1, w_1, x, w_1^2, w_2^2, \cdots, w_{i_2}^2, w_2, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ is the model for $G/(W \cup \{x\})$. If $(w_1, x) \notin E$ then $[(w_1, w_2, x, w_1^1, w_2^1, \cdots, w_{i_1}^1, x, w_1, w_1^2, w_2^2, \cdots, w_{i_2}^2, w_2, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ is the model for $G/(W \cup \{x\})$. Suppose that the model for G/W is $[(w_1, w_2, w_1^1, w_2^1, \cdots, w_{i_1}^1, w_2, w_1^2, w_2^2, \cdots, w_{i_2}^2, w_1, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ where $i_1+i_2+i_3 = 2|W|-4$ (Fig. 3.14b). If $(w_1, x) \in E$ then $[(w_1, w_2, x, w_1^1, w_2^1, \cdots, w_{i_1}^1, w_2, w_1^2, w_2^2, \cdots, w_{i_2}^2, w_1, x, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ is the model for $G/(W \cup \{x\})$. If $(w_1, x) \notin E$ then $[(w_1, w_2, x, w_1^1, w_2^1, \cdots, w_{i_1}^1, w_2, w_1^2, w_2^2, \cdots, w_{i_2}^2, x, w_1, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ is the model for $G/(W \cup \{x\})$.

Used in proof of Fact 22:

w_1 is adjacent to an

occurrence of w_2 and

we would like to add

$x \in M_{W-\{w_2\}}(w_1) \cap A_W^{2+}$

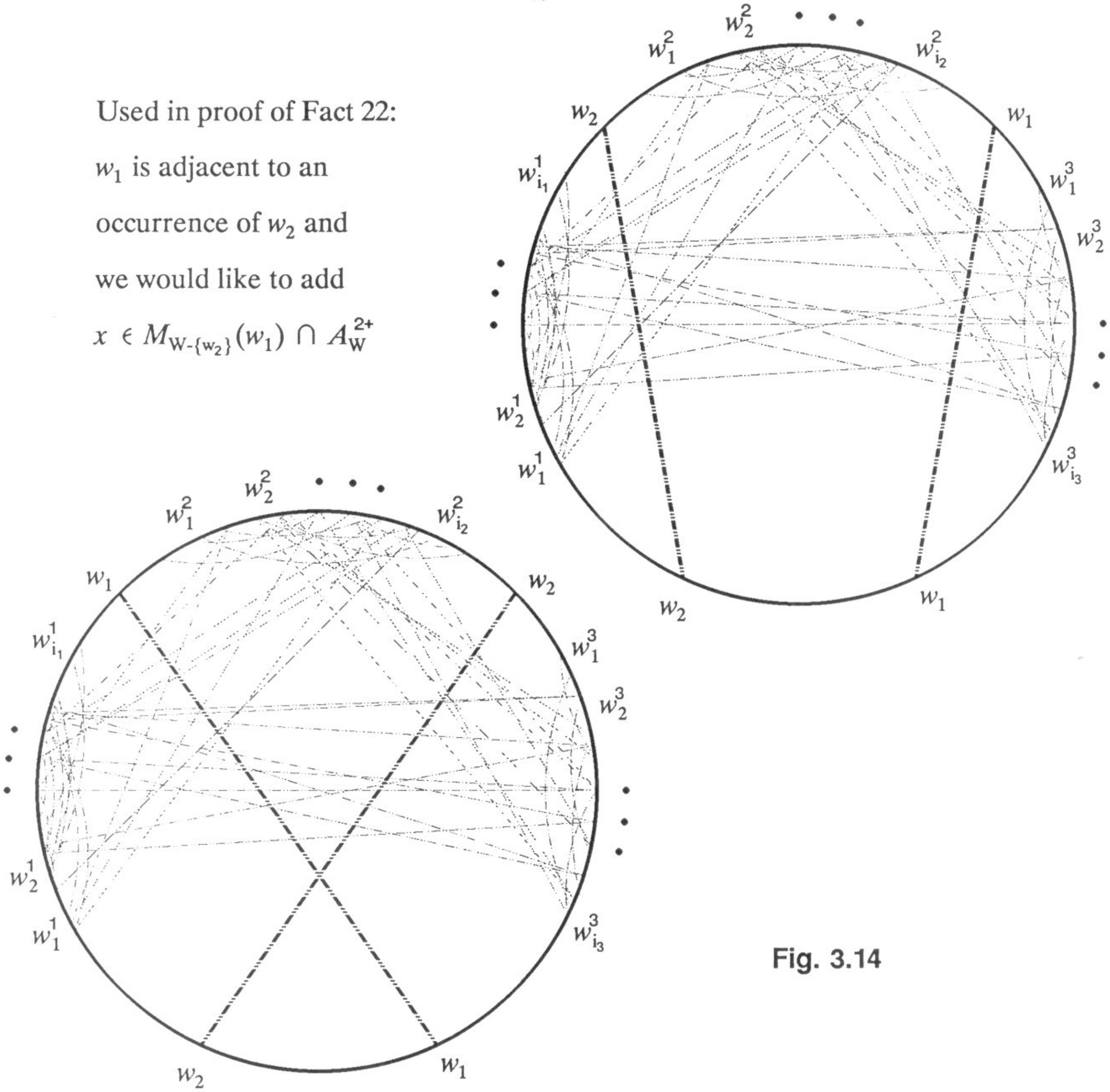


Fig. 3.14

Fact 23: Suppose G/W is a 4-prime circle graph with known model $[S]$, $A = \{p_0, p_1, \cdots, p_{k+1}\} \subseteq W$ such that $G/A = P_{k+2}(p_0, p_1, \cdots, p_{k+1})$ with $k \geq 2$ and $N_A(W-A) = \{p_0, p_{k+1}\}$, and $x \in A_W^{2+}$ but x is adjacent to no vertex in $W-A$. Then it takes O(1) time to determine if $G/(W \cup \{x\})$ is a circle graph and to find the refinement for it, if it is.

Proof of Fact 23: Since G is connected and contains no articulation points, its model $[S]$ is of the form $[(p_0, w_1^1, w_2^1, \cdots, w_{i_1}^1, p_1, p_0, p_2, p_1, p_3, p_2, \cdots, p_{k+1}, p_k, w_1^2, w_2^2, \cdots, w_{i_2}^2, p_{k+1}, w_1^3, w_2^3, \cdots, w_{i_3}^3)]$ (Fig. 3.15). Since there is a set B containing A such that G/B is an

C_n with $n>k+2$, $|N_A(x)| \le 4$ or else $G/(W_1 \cup \{x\})$ is not a circle graph. To specify the refinement of x uniquely, it is sufficient to specify $AI([S'],\{x\})$ where $[S']$ is the model for $G/(W \cup \{x\})$ since $[S' - \{x\}] = [S]$.

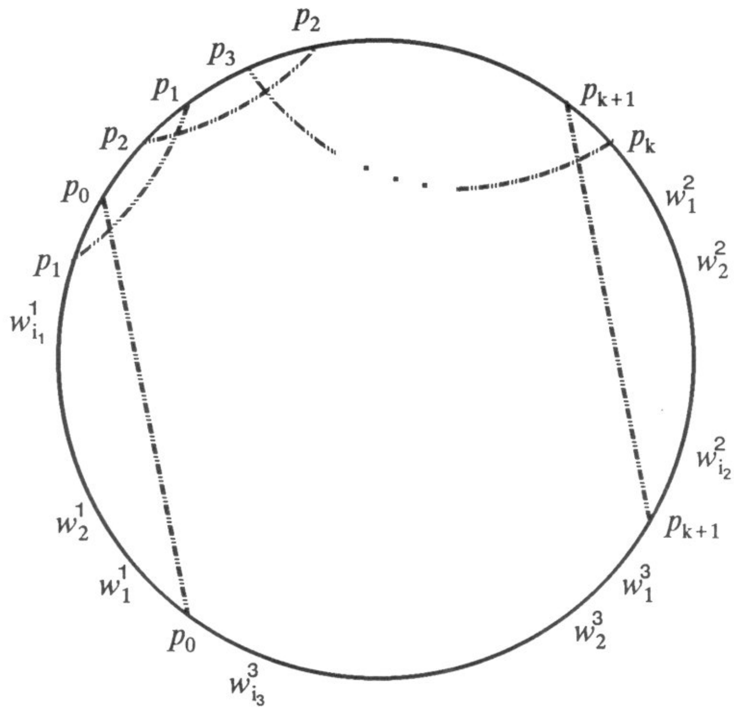


Figure for the proof of Fact 23

Fig. 3.15

Case 1: $N_A(x) = \{p_s, p_t\}$ (where $0 \le s < t \le k+1$ and $s+1 \ne t-1$ since $x \notin M_W(p_{s+1})$). If $0<s, t \le k$ then $AI([S'],\{x\}) = \{\{p_{s-1}, p_{s+1}\}, \{p_{t-1}, p_{t+1}\}\}$. If $0=s, t \le k$ then $AI([S'],\{x\}) = \{\{w_{i_1}^1, p_1\}, \{p_{t-1}, p_{t+1}\}\}$. If $0<s, t=k+1$ then $AI([S'],\{x\}) = \{\{p_{s-1}, p_{s+1}\}, \{p_k, w_1^2\}\}$. If $0=s, t=k+1$ then $AI([S'],\{x\}) = \{\{w_{i_1}^1, p_1\}, \{p_k, w_1^2\}\}$.

Case 2: $N_A(x) = \{p_r, p_s, p_t\}$ (where $r<s<t$). Then since there is a set B containing A such that G/B is an n*−cycle* with $n>k+1$, $G/(W \cup \{x\})$ is a circle graph iff $s=r+1$ or $s=t-1$ (the case $r+1=s=t-1$ cannot happen since $x \notin M_W(p_s)$). If $s=r+1$ and $t<k+1$ then $AI([S'],\{x\}) = \{\{p_s, p_r\}, \{p_{t-1}, p_{t+1}\}\}$. If $s=r+1$ and $t=k+1$ then $AI([S'],\{x\}) = \{\{p_s,$

$p_r\}$, $\{p_k, w_1^2\}\}$. If $r>0$ and $s=t-1$ then $AI([S'],\{x\}) = \{\{p_{r-1}, p_{r+1}\}, \{p_t, p_s\}\}$. If $r=0$ and $s=t-1$ then $AI([S'],\{x\}) = \{\{w_{i_1}^1, p_1\}, \{p_t, p_s\}\}$.

Case 3: $N_A(x) = \{p_r, p_s, p_t, p_u\}$ (where $0 \le r<s<t<u \le k+1$). Then, since there is a set B containing A such that G/B is an n*-cycle* with $n>k+1$, $G/(W \cup \{x\})$ is a circle graph iff $r+1=s$ and $t=u-1$. If it is the case that $r+1=s$ and $t=u-1$ then $AI([S'],\{x\}) = \{\{p_s, p_r\}, \{p_u, p_t\}\}$.

Proof of (1) (from page 78)**:** We will show the following: Suppose (a) G/W is a 4-prime circle graph with a known model, (b) G/W_1 is a circle graph obtained from G/W by one of the rules from Cor. 1, (c) the model for G/W_1 is known, and (d) $x \in A^{2+}{}_{W_1}$ but $x \notin A_W^{2+}$. Then a refinement for the model of $G/(W_1 \cup \{x\})$ or a determination that it doesn't exist can be computed in O(1) time. We have three cases since $x \in A_W^0 \cup A_W^1 \cup M_W$.

Case 1: $x \in A_W^0$. Then $|W_1 - W| \ge 2$.

Case 1.1: Rule 2 of Cor 1 was applied. In this case $N_{W_1}(x) = \{m_1, m_2\}$ and m_1 has an occurrence next-to an occurrenc of m_2 in the model for G/W_1 so Fact 21 applies.

Case 1.2: Rule 3 of Cor 1 was applied. In this case $N_{W_1}(x) \subseteq W_1 - W$ so that Fact 23 applies.

Case 2: $x \in A_W^1(w)$ (for some $w \in W$).

Case 2.1: Rule 1 of Cor 1 was applied so $N_{W_1}(x) = \{w, a\}$. If an occurence of w is next-to an occurence of a in the model for G/W_1 then Fact 21 may be applied. Otherwise, $G/(W_1 \cup \{x\})$ is not a circle graph since if it was, w would be an articulation point of G/W, a contradiction.

Case 2.2: Rule 2 of Cor 1 was applied. Then the occurrences of m_1 are next-to an occurrence, each, of m_2, w_2, w_1, and w_3, and the occurrences of m_2 are next-to an occurrence, each, m_1, w_1, w_2, and w_4 in the model for G/W_1 (where w_3 and w_4 are elements of

$W-\{w_1, w_2\}$). If x is adjacent to exactly one of m_1 or m_2 then $G/(W_1 \cup \{x\})$ is a circle graph if the two elements of $N_{W_1}(x)$ have occurrences next-to each other in the model for G/W_1 (in which case we use Fact 21). Otherwise, $G/(W_1 \cup \{x\})$ is not a circle graph since then we would have w an articulation point of G/W, a contradiction. Thus, suppose x is adjacent to both m_1 and m_2. If $N_{W_1}(x) = \{w_1, m_1, m_2\}$ then $[(x, w_1, m_2, m_1, x, w_2, v_1, v_2, \cdots, v_{2|W|})]$ is the refinement for $G/(W_1 \cup \{x\})$. If $N_{W_1}(x) = \{w_2, m_1, m_2\}$ then $[(w_1, x, m_2, m_1, w_2, x, v_1, v_2, \cdots, v_{2|W|})]$ is the refinement for $G/(W_1 \cup \{x\})$. If either w_3 or w_4 is an element of $N_{W_1}(x)$, then $G/(W_1 \cup \{x\})$ is a circle graph iff $w_3 = w_4$ and in this case $[(w_1, x, m_1, w_3, m_2, x, w_2, u_1, \cdots, u_{2|W|-1})]$ is the refinement for $G/(W_1 \cup \{x\})$. Finally, if $N_W(x) \cap \{w_1, w_2, w_3, w_4\} = \varnothing$ then $G/(W_1 \cup \{x\})$ is not a circle graph since that would imply that the neighbor of x in W is an articulation point in G/W.

Case 2.3: Rule 3 of Cor 1 was applied. In this case the model for G/W_1 is $[(w_3, p_0, w_1, p_1, p_0, p_2, p_1, p_3, p_2, \cdots, p_k, p_{k-1}, w_2, p_k, w_4, v_1, v_2, \cdots, v_{2|W|-4})]$ (Fig. 3.13) and $G/(W_1 \cup \{x\})$ is a circle graph only if $N_W(x) \subseteq \{w_1, w_2, w_3, w_4\}$. If $(x, w_3) \in E$, then $G/(W_1 \cup \{x\})$ is a circle graph iff $(x, p_0) \in E$ and we use Fact 21. If $(x, w_4) \in E$, then $G/(W_1 \cup \{x\})$ is a circle graph iff $(x, p_k) \in E$ and we use Fact 21. If $(x, w_1) \in E$ then by setting $A = \{w_1, p_0, p_1, \cdots, p_{k+1}\}$ and noting that $G/A = P_{k+3}(w_1, p_0, p_1, \cdots, p_{k+1})$, $N_A(W-A) = \{w_1, p_{k+1}\}$, and $N_{W_1}(x) \subseteq A$ we may apply Fact 23. If $(x, w_2) \in E$ then by setting $A = \{w_2, p_{k+1}, p_k, \cdots, p_1, p_0\}$ and noting that $G/A = P_{k+3}(w_1, p_{k+1}, \hat{} \; p_k, \cdots, p_1, p_0)$, $N_A(W-A) = \{w_2, p_0\}$, and $N_{W_1}(x) \subseteq A$ we may apply Fact 23.

Case 3: $x \in M_W(w)$.

Case 3.1: Rule 1 of Cor 1 was applied. Then if an occurrence of a is next-to an occurrence of w, it is unique and Fact 22 may be applied. Otherwise, $G/(W_1 \cup \{x\})$ is not a circle graph.

Case 3.2: Rule 2 of Cor 1 was applied. Then the occurrences of m_1 are next-to occurrences of m_2, w_2, w_1, and w_3, and the occurrences of m_2 are next-to occurrences of m_1, w_1, w_2, and w_4 in the model for G/W_1 (where w_3 and w_4 are elements of $W-\{w_1,w_2\}$). Thus, $G/(W_1 \cup \{x\})$ is a circle graph only if x is W-similar to exactly one of w_1, w_2, w_3, or w_4 (since x is adjacent to at least one of $\{m_1,m_2\}$). If x is W-similar to either w_3 or w_4 then Fact 22 applies. If x is W-similar to w_1, then either x is $(W_1-\{m_1\})$-similar to w_1 or x is $(W_1-\{w_1\})$-similar to m_1. Either way, Fact 22 applies. Similarly, if x is W-similar to w_2, then either x is $(W_1-\{m_2\})$-similar to w_2 or x is $(W_1-\{w_2\})$-similar to m_2. Either way, Fact 22 applies.

Case 3.3: Rule 3 of Cor 1 was applied. In this case the model for G/W_1 is $[(w_3, p_0, w_1, p_1, p_0, p_2, p_1, p_3, p_2, \cdots, p_{k+1}, p_k, w_2, p_{k+1}, w_4, v_1, v_2, \cdots, v_{2|W|-4})]$ (Fig. 3.13). Thus, $G/(W_1 \cup \{x\})$ is a circle graph only if x is W-similar to one of w_1, w_2, w_3, or w_4. If $x \in M_W(w_3)$, then $G/(W_1 \cup \{x\})$ is a circle graph iff $(p_0, x) \in E$ and $(p_i, x) \notin E$ for $i \in \{1,2,\cdots,k+1\}$. Hence, Fact 22 applies. Similarly, if $x \in M_W(w_4)$, then $G/(W_1 \cup \{x\})$ is a circle graph iff $(p_{k+1}, x) \in E$ and $(p_i, x) \notin E$ for $i \in \{0,1,\cdots,k\}$ so Fact 22 applies. Suppose $x \in M_W(w_1)$. If $N_{W_1-W}(x) = \emptyset$ then Fact 22 applies. If $N_{W_1-W}(x) = \{p_s\}$ $(1 \le s \le k+1)$ then $[(w_3, p_0, w_1, p_1, \cdots, p_{s-1}, x, p_{s+1}, \cdots, w_2, p_{k+1}, w_4, \cdots, x, \cdots, v_{2|W|-4})]$ is the model for $G/(W_1 \cup \{x\})$ where the occurrence of w_1 not next-to an occurrence of a_1 nor a_2 is next-to an occurrence of x (Use $p_{k+2} = w_2$). If $N_{W_1-W}(x) = \{p_s, p_{s+1}\}$ $(0 \le s \le k)$ then $[(w_3, p_0, w_1, p_1, \cdots, p_{s+1}, x, p_s, \cdots, w_2, p_{k+1}, w_4, \cdots, x, \cdots, v_{2|W|-4})]$ is the model for $G/(W_1 \cup \{x\})$ where the occurrence of w_1 not next-to an occurrence of a_1 nor a_2 is next-to an occurrence of x. If $N_{W_1-W}(x)$ is different from the above, then G is not a circle graph. Finally, suppose $x \in M_W(w_2)$. This is entirely analagous to situation where $x \in M_W(w_1)$. Thus, if $N_{W_1-W}(x) = \emptyset$ then Fact 22 applies. If $N_{W_1-W}(x) = \{p_s\}$ $(0 \le s \le k)$ then $[(w_3, p_0, \cdots, p_{s-1}, x, p_{s+1}, \cdots, p_{k+1}, p_k, w_2, p_{k+1}, w_4, \cdots, x, \cdots, v_{2|W|-4})]$ is the model for $G/(W_1 \cup \{x\})$ where the occurrence of w_1 not next-to an occurrence of a_1 and a_2 is next-to an occurrence of x (use $p_{-1} = w_1$). If $N_{W_1-W}(x) = \{p_{s-1}, p_s\}$ $(1 \le s \le k+1)$ then $[(w_3, p_0, w_1, \cdots, p_s, x, p_{s-1},$

$\cdots, w_2, p_{k+1}, w_4, \cdots, x, \cdots, v_{2|W|-4})]$ is the model for $G/(W_1 \cup \{x\})$ where the occurrence of w_1 not next-to an occurrence of a_1 and a_2 is next-to an occurrence of x. If $N_{W_1-W}(x)$ is different from the above, then G is not a circle graph. QED (1)

Proof of (2) (from page 78): We show the following: Given $G=(V,E)$ is a circle graph and $W \subseteq V$ induces a 4-prime graph, G/W_1 is obtained from G/W by application of one of the rules of Cor. 1, $x \in A_W^{2+} \cap A^{2+}{}_{W_1}$, the model for G/W_1, $[S]$, is known, and the refinement for the model of $G/(W \cup \{x\})$, $[T]$ is known. Then, it takes O(1) time to calculate the refinement of the model for $G/(W_1 \cup \{x\})$. To prove this assertion, we first show:

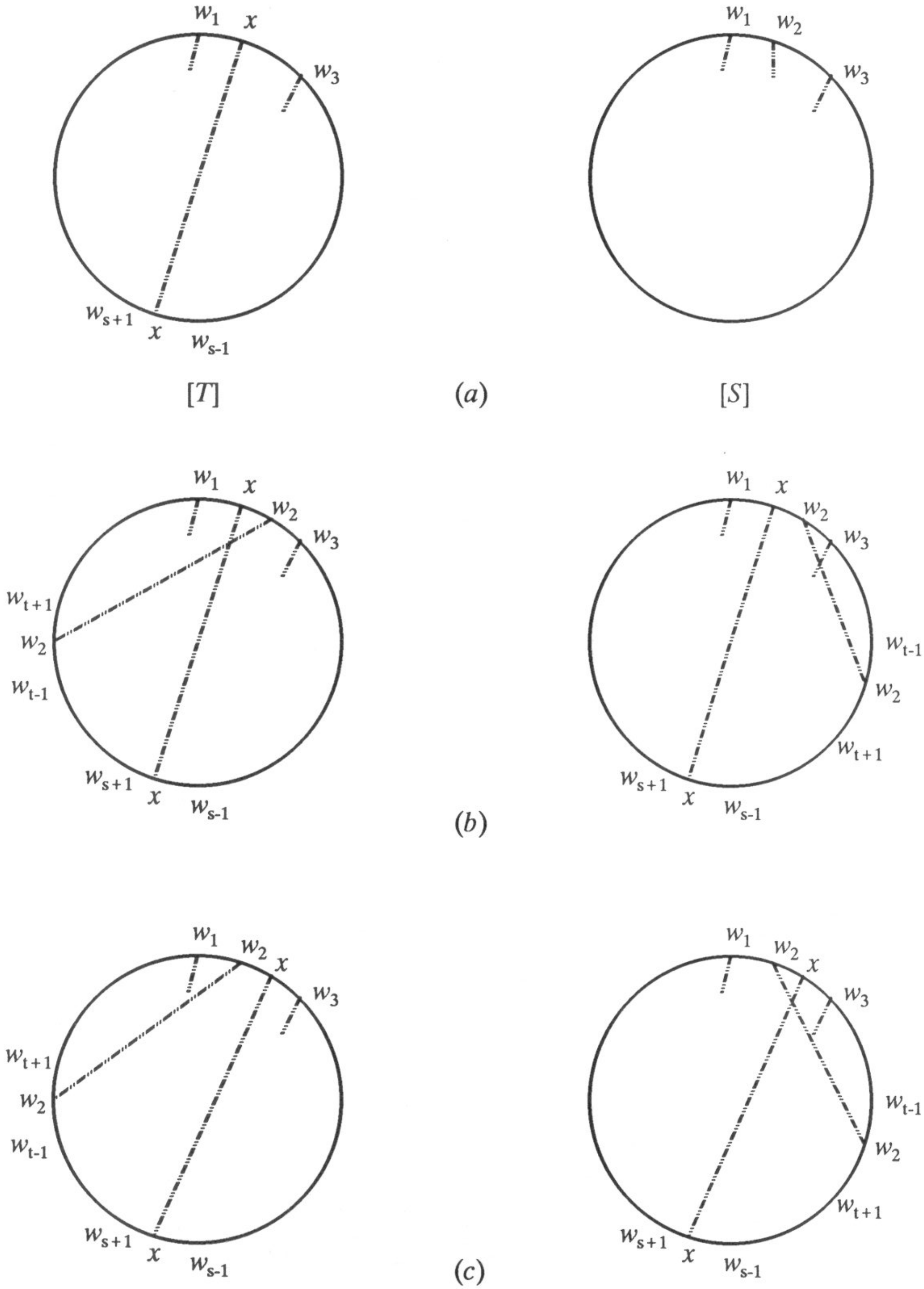
Fact 24: Suppose G/V is a circle graph, G/W is 4-prime, G/W_1 is obtained from G/W by applying a rule from Cor. 1, and $x \in A_W^{2+} \cap A^{2+}{}_{W_1}$. Suppose, also, that the model for G/W_1, $[S]$, and the model for $G/(W \cup \{x\})$, $[T]$, are known. Then:

a) If $AI([S], W_1-W) \cap AI([T], \{x\}) = \varnothing$, the refinement for $G/W_1 \cup \{x\})$ can be computed in O(1) time and is known as a trivial refinement (since the vertices next-to the occurrences of x have remained the same.

b) If there exists $w_2 \in W_1-W$ such that $AI([S], \{w_2\}) \cap AI([T], \{x\}) = \{w_1, w_3\}$ $= AI([S], W_1) \cap AI([T], \{x\})$ (Fig. 3.16a) then the refinement for $G/(W_1 \cup \{x\})$ can be computed in O(1) time and is known as a semi-trivial refinement.

Proof of a): If $AI([T], \{x\}) = \{\{w_1, w_2\}, \{w_3, w_4\}\}$ then the refinement for $G/(W_1 \cup \{x\})$ is the $(W_1 \cup \{x\})$–$2seq$, U, such that $[U-\{x\}]$ is the model for G/W_1 and $AI([U], \{x\}) = AI([T], \{x\})$.

Proof of b): If $[(w_1, w_2, \cdots, w_{2|W_1|})]$ is the model for G/W_1, and $AI([T], \{x\}) = \{\{w_1, w_3\}, \{w_{s-1}, w_{s+1}\}\}$ $(4 < s < 2|W_1|-1)$, and $AI([S], \{w_2\}) = \{\{w_1, w_3\}, \{w_{t-1}, w_{t+1}\}\}$ $(s \neq t$, $4 < t < 2|W_1|-1)$, then if $(w_2, x) \in E$ and $s<t$ or $(w_2, x) \notin E$ and $s>t$ then $[(w_1, x, w_2, \cdots, w_{s-1}, x, w_{s+1}, \cdots, w_{2|W_1|})]$ is the refinement for $G/(W_1 \cup \{x\})$ (Fig. 3.16b). Otherwise $((w_2,$



Proof of Fact 24 (b)

Fig. 3.16

$x) \notin E$ and $s<t$ or $(w_2, x) \in E$ and $s>t$), $[(w_1, w_2, x, \cdots, w_{s-1}, x, w_{s+1}, \cdots, w_{2|W_1|})]$ is the refinement for $G/(W_1 \cup \{x\})$ (Fig. 3.16c).

In proving (2) (from page 78) we are looking for an easy way to specify the model for $G/(W_1 \cup \{x\})$. Since we know the model for G/W_1, it is sufficient to specify $AI([U], \{x\})$. We are now ready to examine the three cases in the proof of (2):

Case 1: Rule 1 of Cor. 1 is used.

In this case, the refinement is either trivial or semitrivial, so we are done.

Case 2: Rule 2 of Cor. 1 is used.

In this event, there are two cases where the refinement is neither trivial nor semitrivial.

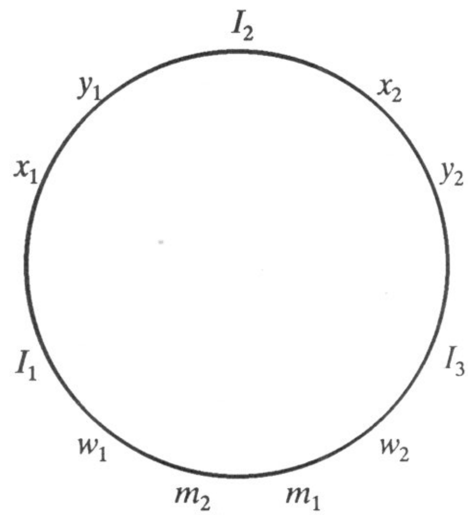
Case 2.1: $AI([T], \{x\}) \cap AI([S], \{m_1, m_2\}) = \{w_1, w_2\}$. Suppose that $AI([T], \{x\}) = \{\{w_1, w_2\}, \{v_1, v_2\}\}$ (Fig. 3.17a). We then have three subcases:

Case 2.1.1: x is adjacent to both or neither of w_1 and w_2. In this event x is adjacent to at least one of m_1 and m_2 if it is not adjacent to w_1 and w_2, and it is not adjacent to both of m_1 and m_2 if it is adjacent to w_1 and w_2 if G is a circle graph. Thus, $AI([U], \{x\}) = \{\{v_1, v_2\}, \{w_1, m_2\}\}$ if x is adjacent to exactly one of w_1 and m_1 and x is adjacent to both or neither of w_2 and m_2. $AI([U], \{x\}) = \{\{v_1, v_2\}, \{m_1, m_2\}\}$ if x is adjacent to exactly one of w_1 and m_1 and x is adjacent to exactly one of w_2 and m_2. $AI([U], \{x\}) = \{\{v_1, v_2\}, \{m_1, w_2\}\}$ if x is adjacent to both or neither of w_1 and m_1 and x is adjacent to exactly one of w_2 and m_2.

Case 2.1.2: x is adjacent w_1 but not to w_2. In this event, if x is adjacent to m_1, then it is adjacent to m_2 since G is a circle graph. Thus, $AI([U], \{x\}) = \{\{v_1, v_2\}, \{w_1, m_2\}\}$ if $(x, m_1) \notin E$ and $(x, m_2) \notin E$. $AI([U], \{x\}) = \{\{v_1, v_2\}, \{m_2, m_1\}\}$ if $(x, m_1) \notin E$ and $(x, m_2) \in E$. $AI([U], \{x\}) = \{\{v_1, v_2\}, \{m_1, w_2\}\}$ if $(x, m_1) \in E$ and $(x, m_2) \in E$.

Case 2.1.3: x is adjacent w_2 but not to w_1. In this event,^ if x is adjacent to m_2, then it is

$\{\{x_1, y_1\}, \{x_2, y_2\}\} =$
$\{\{m_1, w_1\}, \{m_2, w_2\}\}$



[S]

Figure in the proof of (2): Case 2.1

Case 2.1.1 occurs if x is in I_2
Case 2.1.2 occurs if x is in I_1
Case 2.1.3 occurs if x is in I_3

Fig. 3.17a

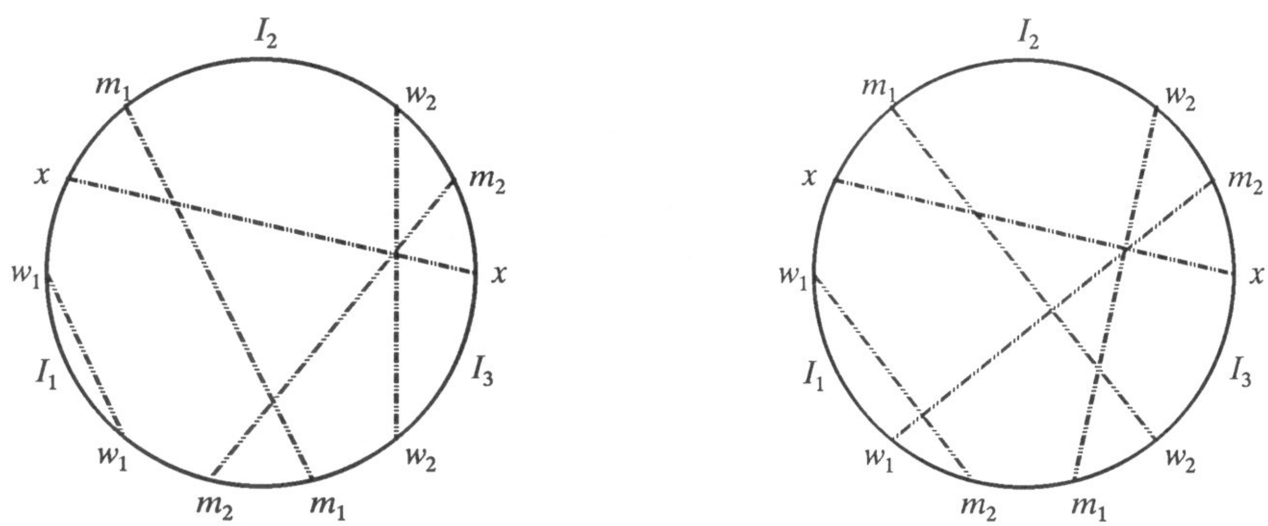


Figure in proof of (2): Case 2.2

Fig. 3.17b

adjacent to m_1 since G is a circle graph. Thus, $AI([U],\{x\}) = \{\{v_1, v_2\}, \{w_1, m_2\}\}$ if $(x, m_1) \in E$ and $(x, m_2) \in E$. $AI([U],\{x\}) = \{\{v_1, v_2\}, \{m_2, m_1\}\}$ if $(x, m_1) \in E$ and $(x, m_2) \notin E$. $AI([U],\{x\}) = \{\{v_1, v_2\}, \{m_1, w_2\}\}$ if $(x, m_1) \notin E$ and $(x, m_2) \notin E$.

Case 2.2: $AI([S],\{m_1, m_2\}) \cap AI([T],\{x\}) = \{\{w_1, v_1\},\{w_2, v_2\}\}$ where v_1, $v_2 \in W - \{w_1, w_2\}$. In this case, if x is adjacent to exactly one of m_1 and w_1 then $\{m_1, w_1\} \in AI([U],\{x\})$. Otherwise, $\{v_1, m_1\} \in AI([U],\{x\})$. Similarly, if x is adjacent to exactly one of m_2 and w_2 then $\{m_2, w_2\} \in AI([U],\{x\})$. Otherwise, $\{v_2, m_2\} \in AI([U],\{x\})$ (see Fig. 3.17b for examples).

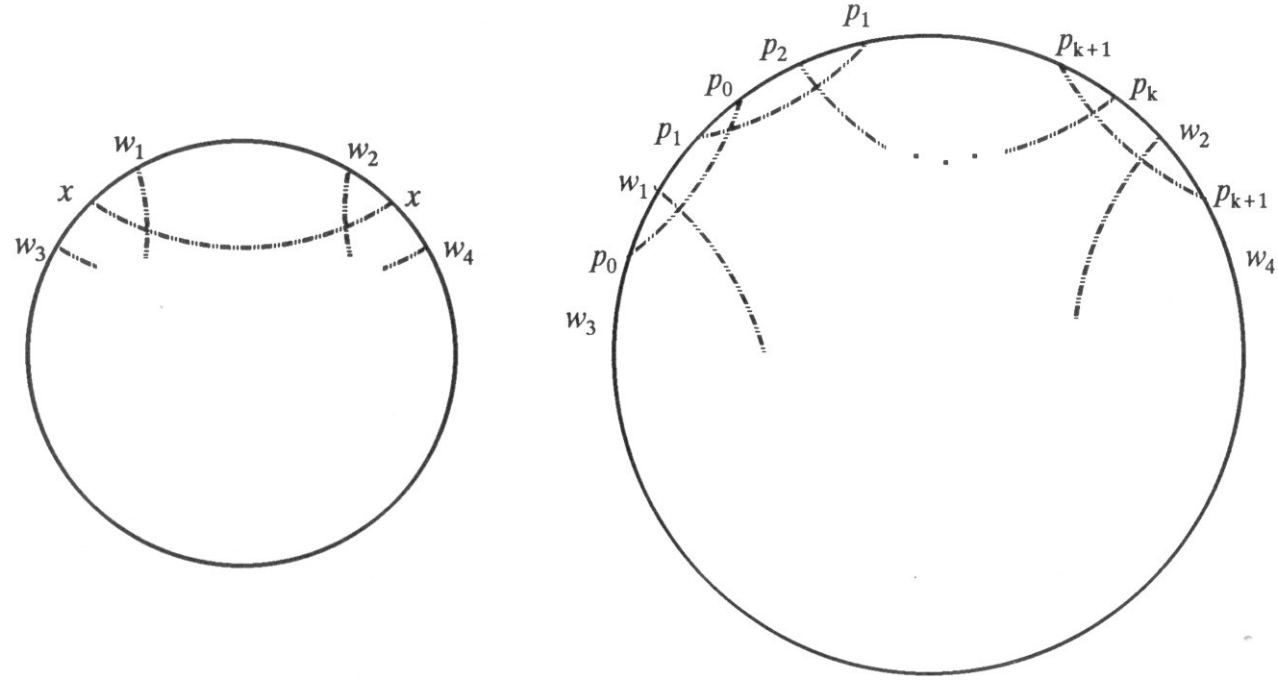
Case 3: Rule 3 of Cor 2 is used. In this event, there are also two cases where the refinement is neither trivial nor semitrivial.

Case 3.1: $AI([T],\{x\}) = \{\{w_1, w_3\}, \{w_2, w_4\}\}$ (Fig. 3.18a) (In this case we have $N_W(x) = \{w_1, w_2\}$). If $(x, a_1) \in E$ then $\{a_1, w_1\} \in AI([U],\{x\})$. Otherwise $\{a_1, w_3\} \in AI([U],\{x\})$. If $(x, a_2) \in E$ then $\{a_2, w_2\} \in AI([U],\{x\})$. Otherwise $\{a_2, w_4\} \in AI([U],\{x\})$.

Case 3.2: $AI([T],\{x\}) = \{\{w_1, w_2\},\{v_1, v_2\}\}$ (Fig. 3.18b). In this case, if $N_{W_1 - W}(x) = \{p_s\}$ $(1 \le s \le k)$ then $AI([U],\{x\}) = \{\{p_{s-1}, p_{s+1}\},\{v_1, v_2\}\}$. If G is a circle graph, the only other possibility is that $N_{W_1 - W}(x) = \{p_s, p_{s+1}\}$ $(1 \le s < k)$, and then $AI([U],\{x\}) = \{\{p_s, p_{s+1}\},\{v_1, v_2\}\}$. QED (2). Hence:

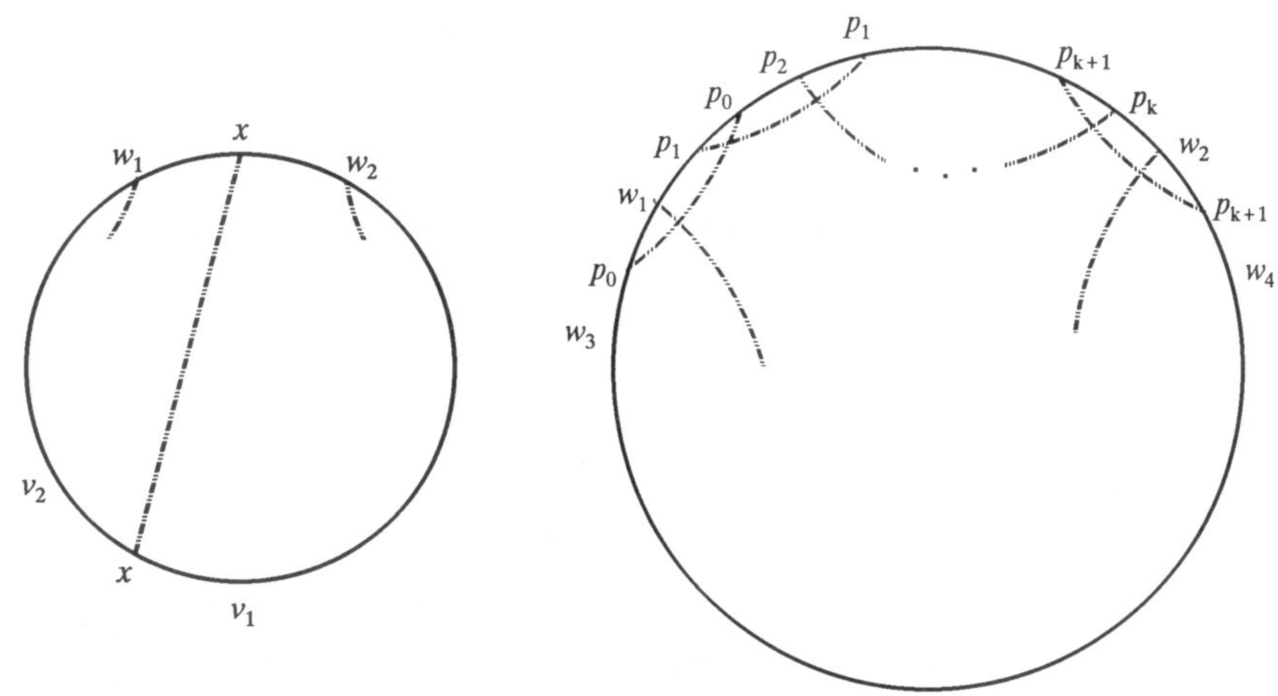
Theorem 6: There is an O(|V| |E|) algorithm to recognize circle graphs.

Note that construction of the model is an $O(|V| \times |V|)$ operation while the selection of the appropriate vertices in each step is an $O(|V| |E|)$ process.



[T] Figure for case 3.1 of (2) [S]

Fig. 3.18a



[T] Figure for case 3.2 of (2) [S]

Fig. 3.18b

4. Permutation Graphs

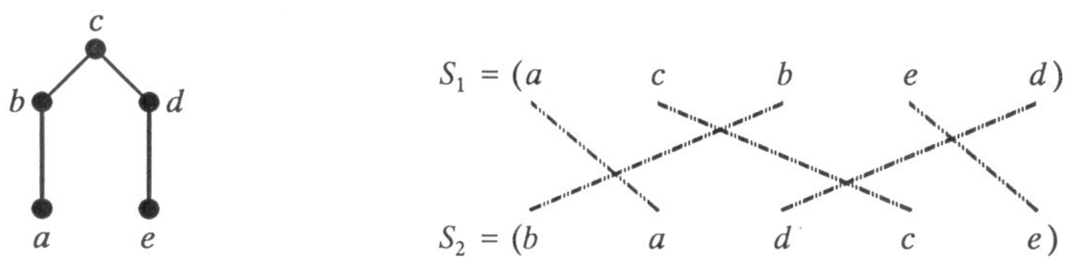
Permutation graphs form a subset of circle graphs. They can be characterized as those circle graphs which remain circle graphs upon the addition of a single new vertex adjacent to all the other vertices. We follow a somewhat more formal development, however, but will quickly utilize the close connection to circle graphs and the results that have been proved in the previous chapters.

Define a V-seq to be any sequence of length $|V|$ such that each element of V is in the sequence. Suppose $S_1 = (s_1^1, s_2^1, \cdots, s_{|V|}^1)$ is a V-seq, and $S_2 = (s_1^2, s_2^2, \cdots, s_{|V|}^2)$ is a V-seq. Define $pos(S_1, s_k^1) = k$. Define $G\ (S_1,\ S_2) = G = (V,E)$ where $(v_1,\ v_2) \in E$ iff $(pos(S_1, v_1) - pos(S_1, v_2)) \times (pos(S_2, v_1) - pos(S_2, v_2)) < 0$ for $v_1, v_2 \in V$. If $G = (V,E) = G(S_1, S_2)$ for two V-seqs S_1 and S_2, then G is a *permutation graph*. The reason for the name permutation graph is that if $G = (V,E) = G(S_1, S_2)$ is a permutation graph then there is a graph $G_1(V_1, E_1)$ with vertex set $V_1 = \{1, 2, \cdots, |V|\}$ which is isomorphic to G such that with $S_1' = (1,2, \cdots, |V|)$ and $S_2' = (pos(S_1, s_1^2), pos(S_1, s_2^2), \cdots, pos(S_1, s^2_{|V|}))$, $G_1 = G(S'_1, S'_2)$. Thus, if an appropriate labelling of G is known, then it is sufficient to specify a permutation, S'_2, to specify G (Fig. 4.1a and b).

There is also a geometric interpretation to a permutation graph. Suppose l_1 and l_2 are two parallel lines with $|V|$ points on each line such that the points on l_1 (read from left to right) are labelled the same as in S_1, and the points on l_2 are labelled the same as in S_2 with the first element of each sequence being the leftmost point on the corresponding line. Connect each pair of identically labelled points (one is always on l_1, the other on l_2). The graph obtained from these line segments is the one whose vertex set consists of the line segments, and where two elements of the vertex set have an edge connecting them iff they intersect (as line segments).

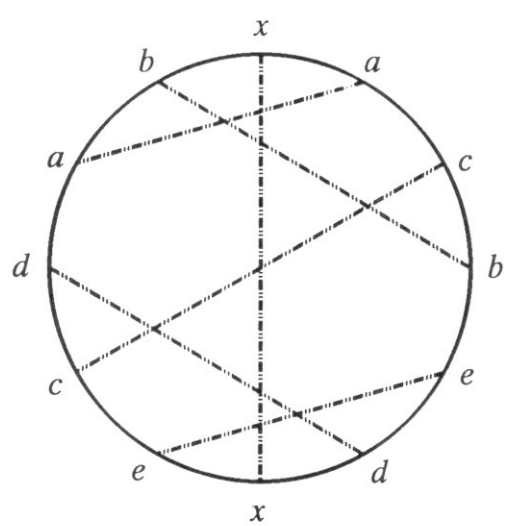
A relationship between permutation graphs and circle graphs can be seen from the geometric representation of permutation graphs. By "deforming" or "bending" the lines l_1 and l_2

of the geometric representation of a permutation graph "into" one another (and bringing the ends in from infinity at the same time) we note that our graph is a circle graph. In fact, if $S_1 = (s_1^1, s_2^1, \cdots, s_n^1)$ and $S_2 = (s_1^2, s_2^2, \cdots, s_n^2)$ then $S = [(s_1^1, s_2^1, \cdots, s_n^1, s_n^2, s_{n-1}^2, \cdots, s_1^2)]$ is a model for the circle graph which represents G. In particular, if $x \notin V$ then $S' = [(x, s_1^1, s_2^1, \cdots, s_n^1, x, s_n^2, s_{n-1}^2, \cdots, s_1^2)]$ is a model for the circle graph such that x is adjacent to each



(a) (b)

A permutation graph along with a model for it



(c)

If x is a new vertex adjacent to all the previousvertices, then this is the circle graph model for the new graph

Fig. 4.1

element of V, and $G(S')/V = G$. Now suppose that $x \in V'$ such that G/V' is a circle graph and $N(x) = V' - \{x\}$. If $S' = [(x, v_1^1, v_2^1, \cdots, v_j^1, x, v_2^1, v_2^2, \cdots, v_k^2)]$ ($j+k+2 = 2|V|$) is the model for G/V', $V_1 = \{v_i^1 | i \in \{1, 2, \cdots, j\}\}$, and $V_2 = \{v_i^2 | i \in \{1,2,\cdots,k\}\}$, then $V_1 = V_2$ since each vertex in $V' - \{x\}$ is adjacent to x. Thus, $G(S' - \{x\})$ is a permutation graph with $S_1 = (v_1^1, v_2^1, \cdots, v^1_{|V|-1})$ and $S_2 = (v^2_{|V|-1}, v^2_{|V|-2}, \cdots, v_1^2)$ (see Fig. 4.1c). The close relationship between circle and permutation graphs is thus:

Cor. 3: Given $G = (V, E)$ and $x \notin V$. G is a permutation graph iff $G'(V \cup \{x\}, E \cup (V \times \{x\}))$ is a circle graph. (G' is G with an extra vertex adjacent to each vertex of G).

This corollary immediately yields the applicable corollaries to the theorems (5 and 6) in chapter 3 for permutation graphs:

Cor. 4: A permutation graph is uniquely representable iff it is 3–*prime*.

Unique representability for permutation graphs means that the only other representations for the given permutation graph are those obtained by reversing both S_1 and S_2, exchanging S_1 and S_2, or both. By combining Cor. 2, Algorithms B2 and C2, and the refinement procedures given for circle graphs, we have an $O(|V| \times |V|)$ recognition algorithm for permutation graphs. [Sp] has previously shown an $O(|V| \times |V|)$ solution to this problem.

5. Conclusions and Further Problems

5.1. Summary

This thesis has shown that under a certain definition of decomposition of graphs, there is a very strong structure within the class of prime graphs - a certain embedded subgraph must exist in each prime graph and an $O(|V|\ |E|)$ algorithm was given to find it. An even stronger result was shown in which it was proved that *each* vertex of a prime graph is contained in a certain embedded subgraph. This result only demonstrated existence and did not provide an algorithm.

The main result was an algorithm (Central Lemma) which will recognize prime graphs in such a way as to find a minimal embedded prime graph (Fig. 1.6) and then incrementally build up to the given graph, at each step maintaining a prime graph and augmenting the graph by one of 4 or 5 rules. This algorithm takes $O(|V|\ |E|)$-time for 4-prime graphs and $O(|V|^2)$-time for 3-prime graphs.

This result is used both in showing the stronger structural theorems (3 and 4) mentioned above and in demonstrating recognition algorithms for two subclasses of graphs, circle and permutation graphs. There is a natural correspondence between 4-prime graphs and circle graphs - circle graphs have a unique representation iff they are 4-prime (Theorem 5). Similarly, the natural correspondence between 3-prime graphs and permutation graphs is that permutation graphs have a unique representation iff they are 3-prime (Cor. 4). Using these relationships, $O(|V| \times |E|)$ and $O(|V|^2)$-time algorithms were found for the recognition of circle and permutation graphs.

5.2. Further Research

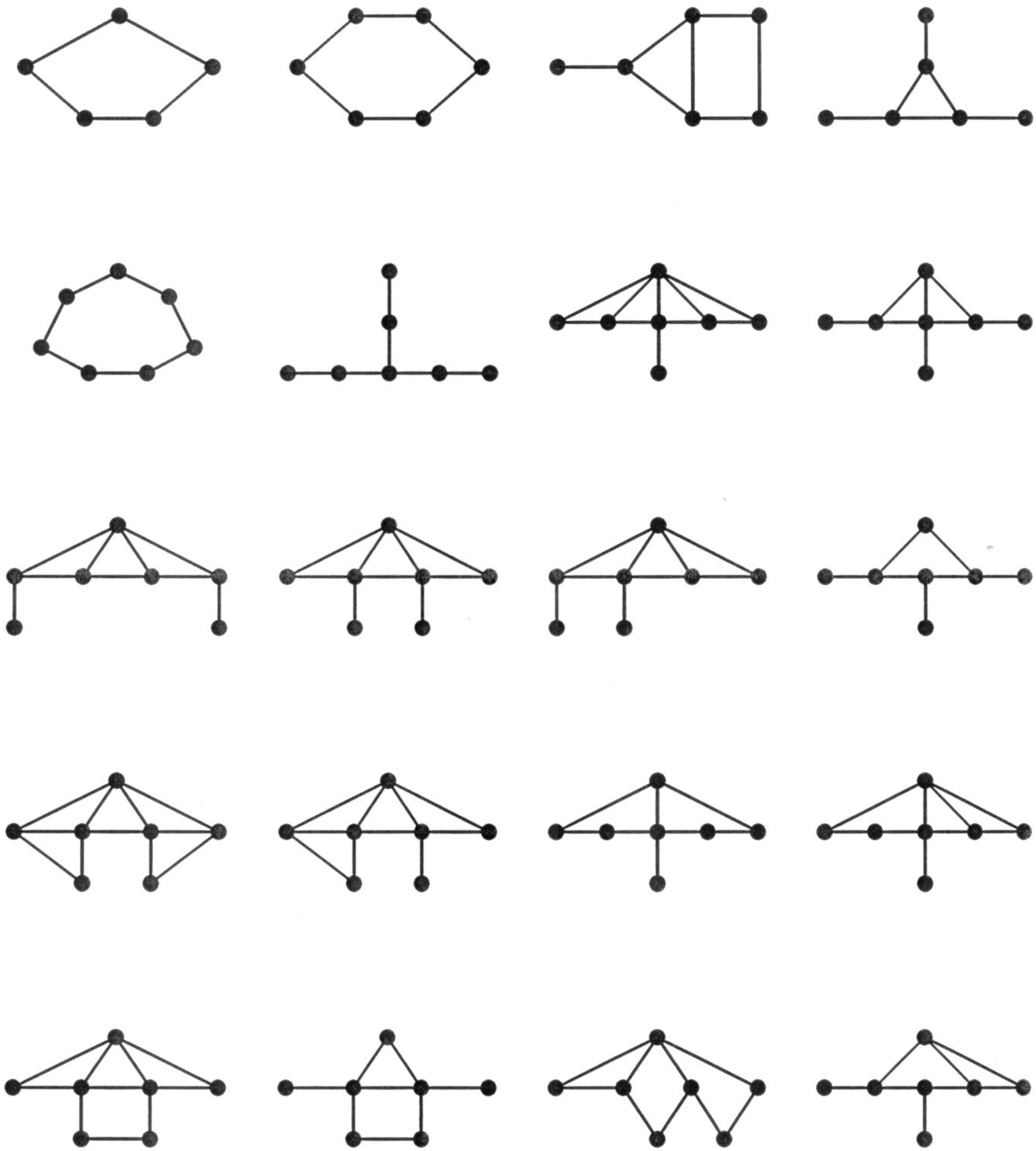
One problem not considered in the body of this dissertation is the structure of non-permutation and non-circle graphs. Suppose a graph is not a permutation graph and there is a vertex whose removal leaves the graph a non-permutation graph. If the vertex is then removed and the process continued until there is no vertex whose removal leaves the graph a non-permutation graph, then the graph arrived at is a minimal non-permutation graph. This graph is clearly 3-prime (by Fact 17 applied to permutation graphs). What may be said about the class of minimal non-permutation graphs (each non-permutation graph contains an embedded minimal non-permutation graph)? In an unpublished work, this author has shown that the set of all minimal non-permutation graphs is that depicted in Fig. 5.1. For each $k \geq 8$ there are exactly 10 graphs in the class if k is odd and 6 if k is even. The proof used, though straightforward, is quite long and tedious. It involves recognizing a non-permutation graph by the method described in the last paragraph in chapter 4 and noting how the method may fail.

The corresponding problem for circle graphs has not been solved. That is, the smallest class of graphs such that an element of this class is embedded in each non-circle graph is not known. Some progress has, however, been made. Given $G = (V, E)$ and $v \in V$, designate by $Lo(cal)Com(plement)_G(v)$ the graph whose vertex set is V and whose edge set is the same except for the following modifications. Neighbors of v in $LoCom_G(v)$ are adjacent iff they are not adjacent in G. Formally, (let $W = N_V(v)$ in G) $LoCom_G(v) = (V, (E \cup W \times W) - (E \cap W \times W))$. Bouchet points out [B1] that G is 4-prime iff $LoCom_G(v)$ is 4-prime for each vertex $v \in V$. This is easy to see: suppose that (V_0, V_1, V_2, V_3) is a 4-decomposing partition of G. If $v \in V_0$, then (V_0, V_1, V_2, V_3) is again a 4-decomposing partition for $LoCom_G(v)$ while if $v \in V_1$, then $(V_0 \cup N_{V_1}(v) - N_{V_0}(v), V_1 \cup N_{V_0}(v) - N_{V_1}(v), V_2, V_3)$ is a 4-decomposing partition for $LoCom_G(v)$ (where the neighborhoods are in the original graph G).

Local complementation has a natural correspondence to circle graphs: $G = (V, E)$ is a circle graph iff $LoCom_G(v)$ is a circle graph for each $v \in V$. It is easy to see this by looking at the circular model for circle graphs. Local complementation is achieved by reversing the subsequence on one side of the chord corresponding to v. Furthermore, from the previous paragraph we have that $G = (V, E)$ is a uniquely representable circle graph iff $LoCom_G(v)$ is too for each $v \in V$.

For a graph $G = (V, E)$ we form $Lo(cal)Class_G = \{H \mid H$ is a graph formed by a sequence of local complementations on $G\}$. for a given size of graphs larger 6, it is easy to see that there are at least 4 distinct *LoClass* 'es since there are 4-prime and non 4-prime circle graphs and there are 4-prime and non 4-prime non-circle graphs. In fact there are exactly two distinct classes of 4-prime circle graphs of size 7. Despite the invariants mentioned above, the local complementation of a minimal non-circle graph need not yield a minimal non-circle graph. The other possibility is that there is an embedded non-circle graph whose size is exactly one less. Each minimal non-circle graph known may be reduced in this fashion to the one non-circle graph of size 6. Unfortunately, it is not known whether there exists a class of minimal non-circle graphs each of whose members is not reducible.

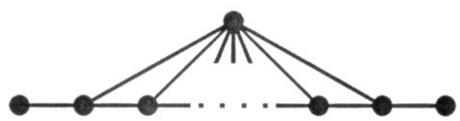
It is straightforward to show that if a graph is 4-prime then by locally complementing it enough times on the appropriate vertices, it is possible to arrive at a graph which may be recognized as 4-prime using only rule 1 of Lemma 1. Another method to determine the set of minimal non-circle graphs would be to analyze how a 4-prime graph may fail to be a circle graph given that only rule 1 is used on some member of a class of minimal non-circle graphs. This seems to be quite a complicated procedure. Perhaps some combination of these methods will yield the desired results.



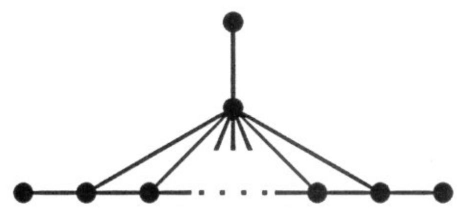
These graphs together with their complements form the set of minimal non-permutation graphs of size less than 8.

Fig. 5.1(a)

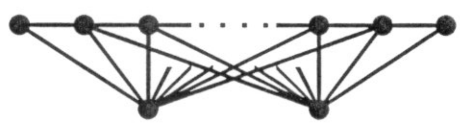
a C_n with $n > 7$



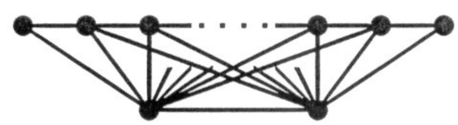
n is odd and greater than 7



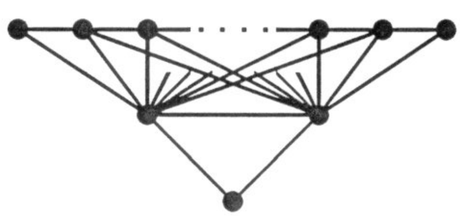
n is odd and greater than 7



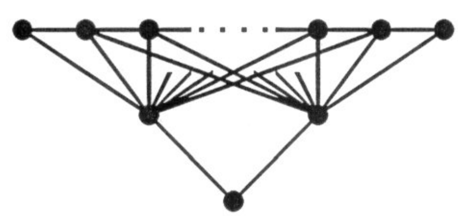
n is odd and greater than 7



n is even and greater than 6



n is odd and greater than 7



n is even and greater than 6

n refers to the total number of vertices in the graph.
These graphs together with their complements form the set of minimal non-permutation graphs of size 8 and larger.

Fig. 5.1(b)

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