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DELAUNAY GRAPHS ARE ALMOST AS GOOD AS COMPLETE GRAPHS

David P. Dobkin
Steven J. Friedman
Kenneth J. Supowit

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Delaunay Graphs are Almost as Good as Complete Graphs †

David P. Dobkin
Steven J. Friedman
Kenneth J. Supowit

Dept. of Computer Science
Princeton University
Princeton, NJ 08544

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Abstract.

Let S be any set of N points in the plane and let $\mathrm{DT}(S)$ be the graph of the Delaunay triangulation of S. For all points a and b of S, let $\mathrm{d}(a,b)$ be the Euclidean distance from a to b and let $\mathrm{DT}(a,b)$ be the length of the shortest path in $\mathrm{DT}(S)$ from a to b. We show that there is a constant c ($\leq \frac{1+\sqrt{5}}{2}\pi \approx 5.08$) independent of S and N such that

$$\frac{\mathrm{DT}(a,b)}{\mathrm{d}(a,b)} < c.$$

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1 Introduction

A major goal in the study of data structures is the discovery of structures which are succinct and efficient. The aim is to find a structure requiring a small amount of space (typically linear) which can then be used for efficient computation. For example, search trees on N data items are built in time $N \log N$ to store data in space N allowing for searches in time $\log N$. In this case, the data structure is used to represent all of the data present. In other situations, efficiency is built into a data structure by not storing all of the data, choosing instead to represent a subset which accurately characterizes the information. For example, consider a depth first spanning tree for a graph. In this case, the backbone of the graph is represented within the data structure and additional back-edges appear in an additional structure. In this paper, we extend the problem one step further. Our goal is to find a linear space data structure which represents a quadratic amount of information effectively.

Our main results concern the use of the Delaunay triangulation of a set of points in the plane as an approximation to the complete graph connecting these points. In particular, we study the question:

> Let S be a set of N points in the plane and let $\mathrm{DT}(S)$ be the graph corresponding to the Delaunay triangulation of S. If a and b are vertices of S, what is the maximum value of the ratio of the shortest path connecting a and b in $\mathrm{DT}(S)$ to the (Euclidean) distance between a and b?

We show that for all N and all point sets S, this ratio is bounded above by a constant. In what follows, we show that this constant is at most $\frac{1+\sqrt{5}}{2}\pi \approx 5.08$.

This problem has been previously studied by Chew [Ch86] who showed that if $DT_1(S)$ is the Delaunay triangulation in the L_1 norm then the ratio of shortest distances in $DT_1(S)$ to straight line distances is bounded above by $\sqrt{10} \approx 3.16$. He gives a lower bound of $\pi/2$ on the ratio we consider here but is unable to prove an upper bound. We conjecture that the true bound is closer to $\pi/2$ than to $\frac{1+\sqrt{5}}{2}\pi$; we suspect that the appearance of the "golden ratio" here is an artifact of our proof technique rather than something inherently related to Delaunay triangulations. Chew also argues based upon applications to motion planning, polygon visibility and extensions of Voronoi diagrams/Delaunay triangulations that the problem which we solve is of significance.

In a recent paper, Sedgewick and Vitter [SV86] studied the problem of finding shortest distances within random Euclidean graphs. They describe applications to transportation problems, AI searches and VLSI routing that arise from problems similar to the one we study here. Further, they state our problem as an interesting problem and suggest that it might be possible to find shortest paths using the Euclidean Delaunay graph in sublinear time.

In addition to the applications mentioned in these papers, our result has potential application to heuristics for finding fast traveling salesman tours and related problems arising from finding shortest distances. Furthermore, the approach studied here represents a paradigm that deserves further study by graph theorists.

In Section 2 we derive our main result from a number of lemmas, which are proved in Section 3.

2 The main result

We begin with (informal) definitions of the Voronoi diagram and the Delaunay triangulation. The *Voronoi diagram* for a set S of N points in the plane is a partition of the plane into regions, each containing exactly one point in S, such that for each point $p \in S$, every point within its corresponding region (denoted $\mathrm{Vor}(p)$) is closer to p than to any other point of S. The boundaries of these regions form a planar graph. The *Delaunay Triangulation* of S is the straight-line dual of the Voronoi diagram for S; that is, we connect a pair of points in S if and only if they share a Voronoi boundary. Under the standard assumption that no four points of S are co-circular, the Delaunay Triangulation is indeed a triangulation [PS85]; we denote its corresponding graph by $\mathrm{DT}(S)$.

For the remainder of this section, fix points $a, b \in S$; we will construct a path in $\mathrm{DT}(S)$ that is not too long in relation to $\mathrm{d}(a, b)$. Assume for simplicity that a and b lie on the

x-axis, with $\mathrm{x}(a) < \mathrm{x}(b)$ (we denote the coordinates of a point q in the plane by $\mathrm{x}(q)$ and $\mathrm{y}(q)$, respectively). We refer to members of S alternatively as points or vertices, and to edges of DT(S) as edges or line segments, as the context indicates.

Our original idea for the path was simply to use the vertices $a = b_0, b_1, \ldots, b_{m-1}, b_m = b$ corresponding to the sequence of Voronoi regions traversed by walking from a to b along the x-axis (see Fig. 1, where $m = 4$). In general, we refer to the DT path constructed in this way between some z and z' in S as the *direct DT path* from z to z'. Let p_i denote the point on the x-axis that also lies on the boundary between Vor(b_{i-1}) and Vor(b_i), for $i = 1, 2, \ldots, m$. The definition of the Voronoi diagram immediately gives that p_i is the center of a circle C_i passing through b_{i-1} and b_i but containing no points of S in its interior.

Two simple properties of direct DT paths are:

Lemma 1: $\mathrm{x}(b_0) \leq \mathrm{x}(b_1) \leq \cdots \leq \mathrm{x}(b_m)$.

Lemma 2: For all $i, 0 \leq i \leq m$, b_i is contained within, or on the boundary of, circle(a, b) (by which we denote the circle with a and b diametrically opposed).

Note in Fig. 1 that all the b_i happen to be above the x-axis (i.e. $\mathrm{y}(b_i) > 0$ for all $0 \leq i \leq m$. In such cases, we say that the direct path between the two points is *one-sided.* One-sided paths are fortuitous for our purposes, because the ratio of the path length to the Euclidean distance is at most $\pi/2$; this is a simple consequence of Lemma 1 above and the following:

Lemma 3: Let $D_1, D_2, \ldots, D_k$ be circles all centered on the x-axis such that $D = \bigcup_{1 \leq i \leq k} D_i$ is connected. Then boundary(D) has length at most $\pi \cdot (x_r - x_\ell)$, where x_ℓ and x_r are the least and greatest $x-$coordinates of D, respectively (see Fig. 2).

Lemma 3 applies to the one-sided paths because the half of boundary(C) (where C is defined as $\bigcup_{1 \leq k \leq m} C_k$) that lies above the x-axis has length at least as great as the path itself (because the b_i are monotonic in x).

The trouble with this approach is that the path is not necessarily even close to being one-sided; the path may zig-zag across the x-axis (as is illustrated in Fig. 3) $\Theta(N)$ times.

Our modified approach, then, is to try to stay above the x-axis. Should the direct path dip below the x-axis, we determine how costly the dip will be. If dipping below is not too expensive (in a sense defined below) then we follow the direct path below the x-axis and then back up. Otherwise, we construct a shortcut between the two points above the x-axis. Most of the proof consists of showing that the shortcut is not too long. The exact path we take is made more precise in the proof of the following:

Theorem: There exists a DT path from a to b of length at most $\leq \frac{1+\sqrt{5}}{2}\pi \cdot \mathrm{d}(a,b)$

Proof:

We present an algorithm for constructing a DT path from $a = b_0$ to $b = b_m$, and then analyze the length of the path it produces. Assume that the path so far has brought us to some b_i such that (1) $\mathrm{y}(b_i) \geq 0$ (initially, $i = 0$), (2) $i < m$ (meaning we're not finished), and (3) $\mathrm{y}(b_{i+1}) < 0$. Thus the direct path would dip below the x-axis for a while after b_i. Let j be the least number greater than i such that $\mathrm{y}(b_j) \geq 0$ (e.g. in Fig. 4, if $i = 2$ then $j = 4$). Let T denote the path along the boundary of C clockwise from b_i to b_j. Let w denote the length of the projection of T onto the x-axis (thus $w = \mathrm{x}(b_j) - \mathrm{x}(b_i)$). Define $h = \min\{\mathrm{y}(q) : q \text{ lies on } T\}$. Now if $h \leq w/4$ then continue along the direct path to b_j (i.e. use edges $b_i b_{i+1}$, $b_{i+1} b_{i+2}$, ..., $b_{j-1} b_j$). Otherwise we take a shortcut as follows. Construct the lower convex hull $b_i = z_0$, z_1, z_2, ..., $z_n = b_j$ of the set

$$\{q \in S : \mathrm{x}(b_i) \leq \mathrm{x}(q) \leq \mathrm{x}(b_j) \text{ and } \mathrm{y}(q) \geq 0 \text{ and } q \text{ lies under } b_i b_j\}$$

(see Fig. 5). Note that these convex hull edges are certainly not on the direct DT path from a to b. Now the shortcut consists of taking the direct DT path from z_k to z_{k+1} for each $0 \leq k \leq n-1$. The key fact (proved in Section 3) is:

Lemma 4: Let $z_k z_{k+1}$ be an edge of the lower convex hull described above. Then the direct DT path from z_k to z_{k+1} is one-sided.

Next we analyze the length of the path produced by this algorithm. When proceeding from b_i to b_j, let t denote the length of T. If $h \leq w/4$ then let q_0 be the point of T with least y-value (see Fig. 6), let t_i denote the length of the portion of T from b_i to q_0, and t_j the length of the portion of T from q_0 to b_j (thus $t_i + t_j = t$). Let w_i and w_j denote the lengths of the projections of those two portions of T, respectively (thus $w_i + w_j = w$). Then the path we take (i.e. no shortcuts) has length at most

$$\begin{aligned} t + 2(y(b_i) + y(b_j)) &= t + 2(2h + (y(b_i) - h) + (y(b_j) - h)) \\ &\leq t + 2(\frac{w}{2} + (y(b_i) - h) + (y(b_j) - h)) \\ &= t + 2(\frac{w_i}{2} + (y(b_i) - h) + \frac{w_j}{2} + (y(b_j) - h)) \\ &\leq t + 2(\frac{\sqrt{5}}{2}t_i + \frac{\sqrt{5}}{2}t_j) = t(1 + \sqrt{5}) \end{aligned}$$

The last inequality follows from the (easily proved) fact that

$$\frac{a}{2} + b \leq \frac{\sqrt{5}}{2}c$$

whenever a and b are the legs of a right triangle with hypotenuse c.

On the other hand if $h > w/4$ then we take the shortcut, which has length at most

$$\sum_{k=0}^{n-1} \text{length of one-sided path from } z_k \text{ to } z_{k+1}$$

(by Lemma 4) which is $\leq \sum_{k=0}^{n-1} d(z_k, z_{k-1})\pi/2 \leq t\pi/2$. (by Lemma 3). Hence in either case, the distance we travel in getting from b_i to b_j is at most $(1+\sqrt{5})t$. Therefore summing over all such trips b_i to b_j as well as the trips (for which we travel at most t units) where the direct DT path from a to b stays completely above the x-axis, we get (by Lemma 3) a total path length of at most $d(a, b)\frac{1+\sqrt{5}}{2}\pi$. □

3 Proofs of the lemmas

Proof of Lemma 1: The perpendicular bisector of b_i and b_{i+1} contains p_i. Point b_{i+1} lies to the right of this bisector, and b_i lies to the left; hence $\mathrm{x}(b_i) \leq \mathrm{x}(b_{i+1})$. □

Proof of Lemma 2: Let c denote the midpoint of segment ab; let k be such that c lies in the Voronoi region of b_k. Then

$$\mathrm{d}(b_0, c) \geq \mathrm{d}(b_1, c) \geq \ldots \geq \mathrm{d}(b_k, c)$$

and

$$\mathrm{d}(b_k, c) \leq \mathrm{d}(b_{k+1}, c) \leq \ldots \leq \mathrm{d}(b_m, c)\,. \qquad \square$$

Proof of Lemma 3: By induction on k. The claim is easy if $k = 1$; so let $k \geq 2$ and assume it for $k - 1$. Let q_1 and q_4 denote the leftmost and rightmost points of the D_k, respectively (see Fig. 7), and assume without loss of generality that $q_4 = x_r$. Let q_2 be the rightmost point at which D_k intersects another circle D_j (thus $j < k$); let q_3 be the rightmost point of D_j. We can assume that D_k does not entirely contain any circle D_i $(i \neq k)$, since otherwise D_i would not contribute to boundary(D) and hence the induction would be trivial. Denote by α_1 (α_2) the length of the arc on circle D_k clockwise from q_1 to q_2 (resp. q_2 to q_4). Let α_3 be the length of the arc on circle D_j clockwise from q_2 to q_3. Finally, let $\alpha_4 = (\pi/2)(\mathrm{x}(q_3) - \mathrm{x}(q_1))$ and let $\alpha_5 = (\pi/2)(\mathrm{x}(q_4) - \mathrm{x}(q_3))$. Then a simple convexity argument shows that

$$\alpha_1 + \alpha_3 \geq \alpha_4\,.$$

Also, we have

$$\alpha_4 + \alpha_5 = \alpha_1 + \alpha_2\,.$$

Hence

$$\alpha_1 + \alpha_3 + \alpha_5 \geq \alpha_4 + \alpha_5 = \alpha_1 + \alpha_2\,,$$

implying $\alpha_3 + \alpha_5 \geq \alpha_2$. Therefore, denoting the length of the boundary of D by $\mathrm{bd}(D)$, we have

$$\begin{aligned}\mathrm{bd}(D) &\le \mathrm{bd}(\mathrm{circle}(q_3, q_4) \cup \bigcup_{1\le i\le k-1} D_i) \\ &\le \mathrm{bd}(\mathrm{circle}(q_3, q_4)) + \mathrm{bd}\left(\bigcup_{1\le i\le k-1} D_i\right) \\ &\le \pi(x_r - \mathrm{x}(q_3)) + \pi(\mathrm{x}(q_3) - x_\ell) \text{ (by the inductive hypothesis)} \\ &\le \pi(x_r - x_\ell).\end{aligned}$$

□

Proof of Lemma 4:

By Lemma 2, the direct DT path from z_k to z_{k+1} lies entirely within circle(z_k, z_{k+1}). We now show that there are no points of S within the lower semicircle of circle(z_k, z_{k+1}), so the path must be one-sided.

Let q be an arbitrary point in this lower semicircle; we must show $q \notin S$. If $\mathrm{x}(b_i) \le \mathrm{x}(q) \le \mathrm{x}(b_j)$ and $\mathrm{y}(q) \ge -h$ (i.e. q lies in region R_1 in Fig. 5) then we claim $q \notin S$. To see this, note that if $\mathrm{y}(q) \ge h$ then it lies outside the lower convex hull; whereas if $-h < \mathrm{y}(q) < h$ then q lies in the interior of $\bigcup_{i\le k\le j} C_k$.

We next show that $\mathrm{y}(q) > -h$ (that is, $q \notin R_2$). Assume without loss of generality that $\mathrm{y}(z_k) \le \mathrm{y}(z_{k+1})$. Since $z_k \in S$ it must lie directly above some point of T, since the area below T and above the x-axis is contained in C and therefore contains no members of S. Therefore $\mathrm{y}(z_k) \ge h > w/4$. Let z' be the point with coordinates $(\mathrm{x}(z_{k+1}), \mathrm{y}(z_k))$. Let c and c' denote the midpoints of segments $z_k z_{k+1}$ and $z_k z'$, respectively. Then $\mathrm{y}(c') > w/4$. That $q \in \mathrm{circle}(z_k, z')$ follows from $q \in \mathrm{circle}(z_k, z_{k+1})$ and $\mathrm{y}(q) \le \mathrm{y}(z_k) = \mathrm{y}(z')$. Furthermore, $\mathrm{x}(z_{k+1}) - \mathrm{x}(z_k) \le w$, since by extending $z_k z_{k+1}$ on both sides we encounter points on T and since T is connected (and hence the projection of T onto the x-axis is at least as long as the projection of $z_k z_{k+1}$ onto the x-axis). Therefore radius(circle(z_k, z')) $\le w/2$. Hence

$$\mathrm{y}(q) \ge \mathrm{y}(c') - \mathrm{radius}(\,\mathrm{circle}(z_k, z')\,) > w/4 - w/2 = -w/4\,.$$

Finally, we assume $\mathrm{x}(q) > \mathrm{x}(b_j)$ (hence $q \in R_4$) (the case $\mathrm{x}(q) < \mathrm{x}(b_i)$, that is $q \in R_3$, is handled analogously). We will show that q lies in the interior of C_j, implying $q \notin S$. Let x_ℓ be the leftmost point of intersection of circle C_j with the line $y = h$. Let x_r be the rightmost point of intersection of C_j with the line $y = -h$. Let ℓ denote the line that passes through z_{k+1} perpendicular to segment $z_k z_{k+1}$, and let ℓ' be the line containing b_j and x_r. Note that both ℓ and ℓ' both must have negative slopes. Clearly the entire circle(z_k, z_{k+1}) lies below ℓ and in particular so does q. We claim that this implies that q lies below ℓ' as well. To see this, first note that our assumption $\mathrm{y}(z_k) \leq \mathrm{y}(z_{k+1})$ implies $\mathrm{y}(z_{k+1}) \leq \mathrm{y}(b_j)$, and hence line ℓ intersects the line $x = \mathrm{x}(b_j)$ below b_j. Therefore it suffices to show that slope$(\ell) \leq$ slope(ℓ') (recall that both are negative). The monotonicity of slopes in the lower convex hull gives slope$(z_k z_{k+1}) \leq$ slope$(x_\ell b_j)$. Therefore since ℓ and ℓ' are perpendicular to $z_k z_{k+1}$ and $x_\ell b_j$, respectively, (the latter is because x_ℓ and x_r are diametrically opposed on C_j), we have slope$(\ell) \leq$ slope(ℓ'). Thus q indeed lies below ℓ'; hence since q is in R_4 it must also be in C_j and therefore not in S. □

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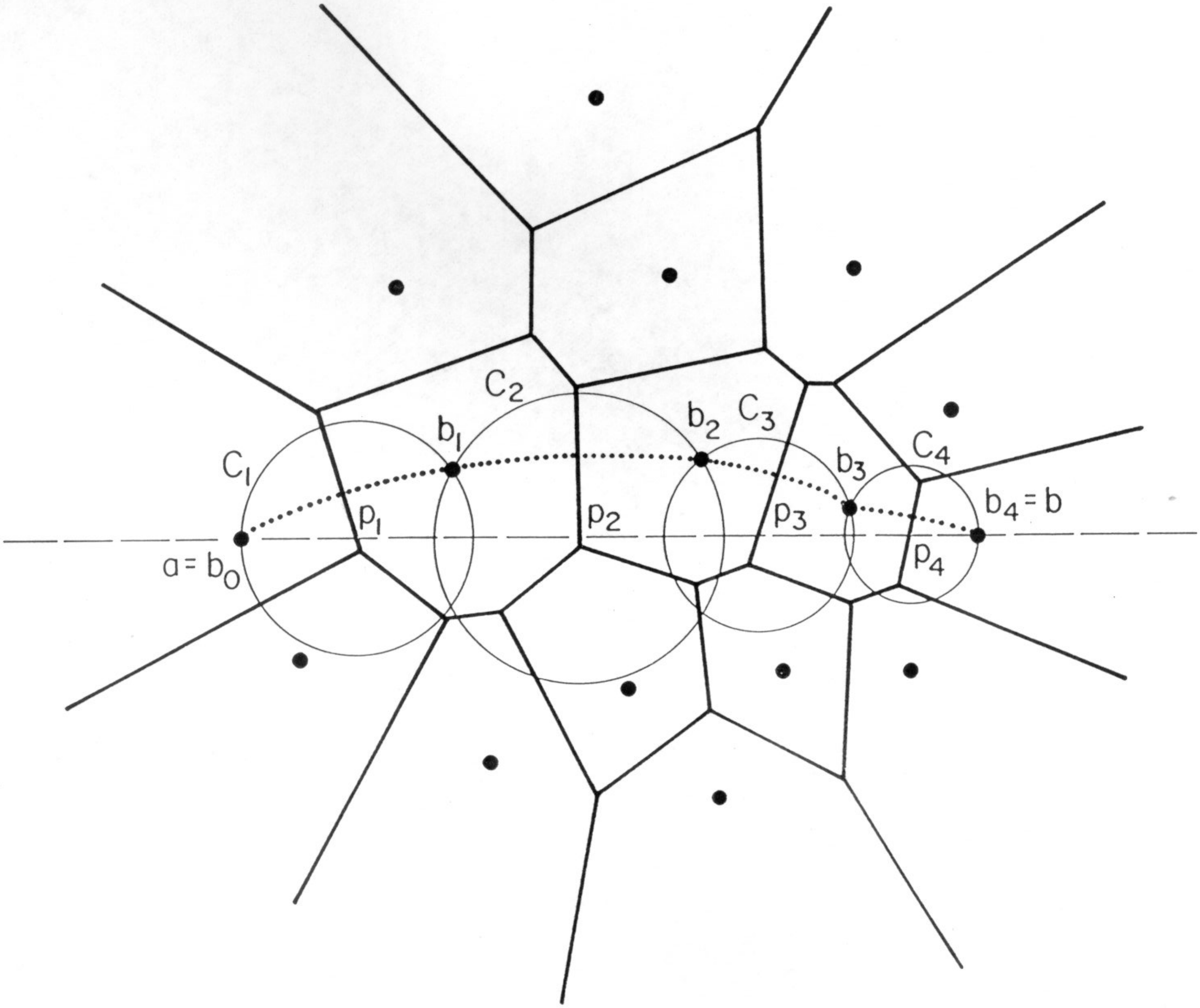

Figure 1

The Voronoi Diagram is shown in solid line,
and the direct DT path between a and b in dotted line.

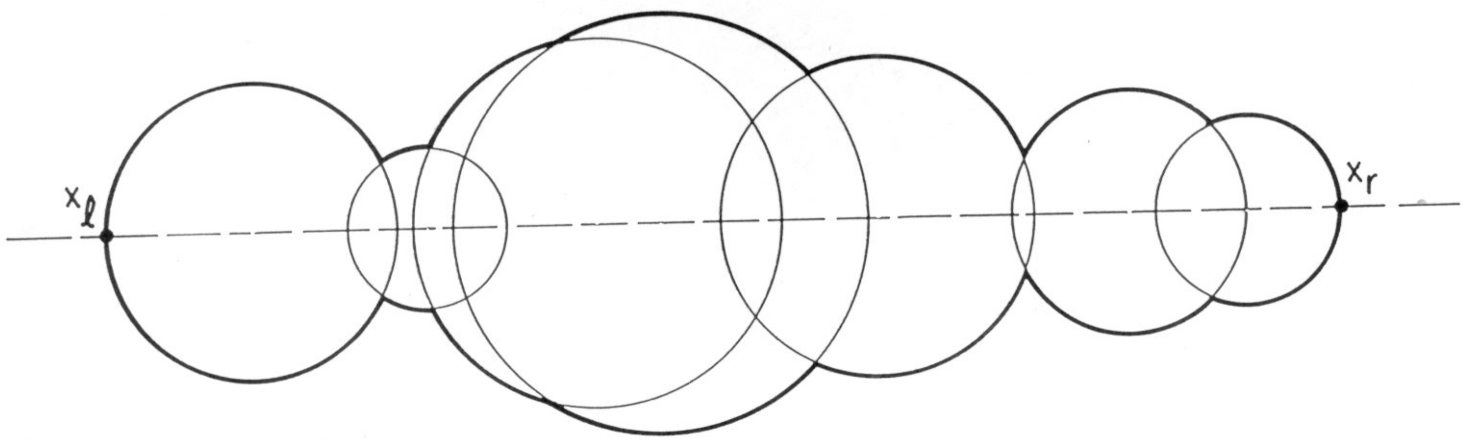

Figure 2

Illustration for Lemma 3.

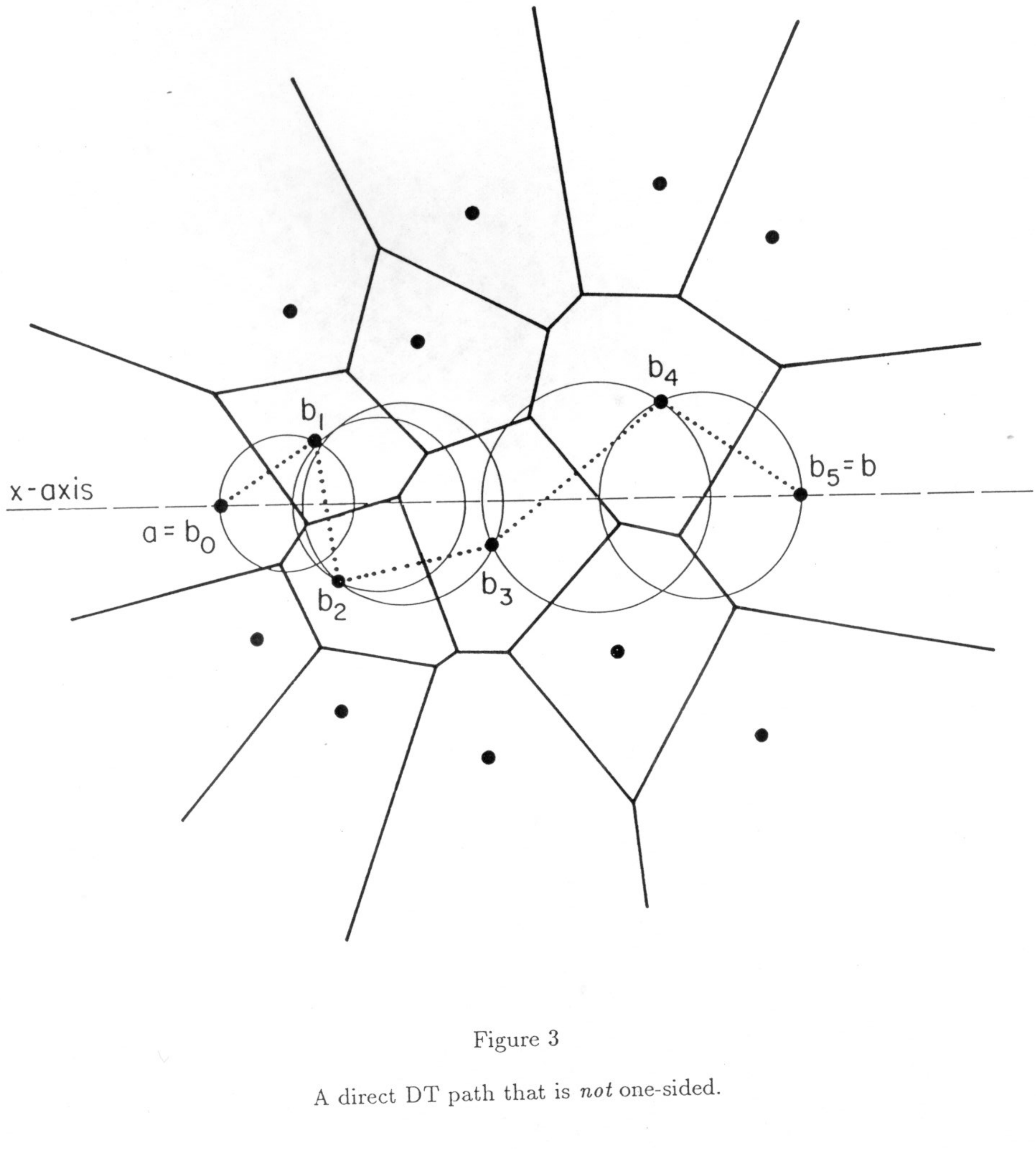

Figure 3

A direct DT path that is *not* one-sided.

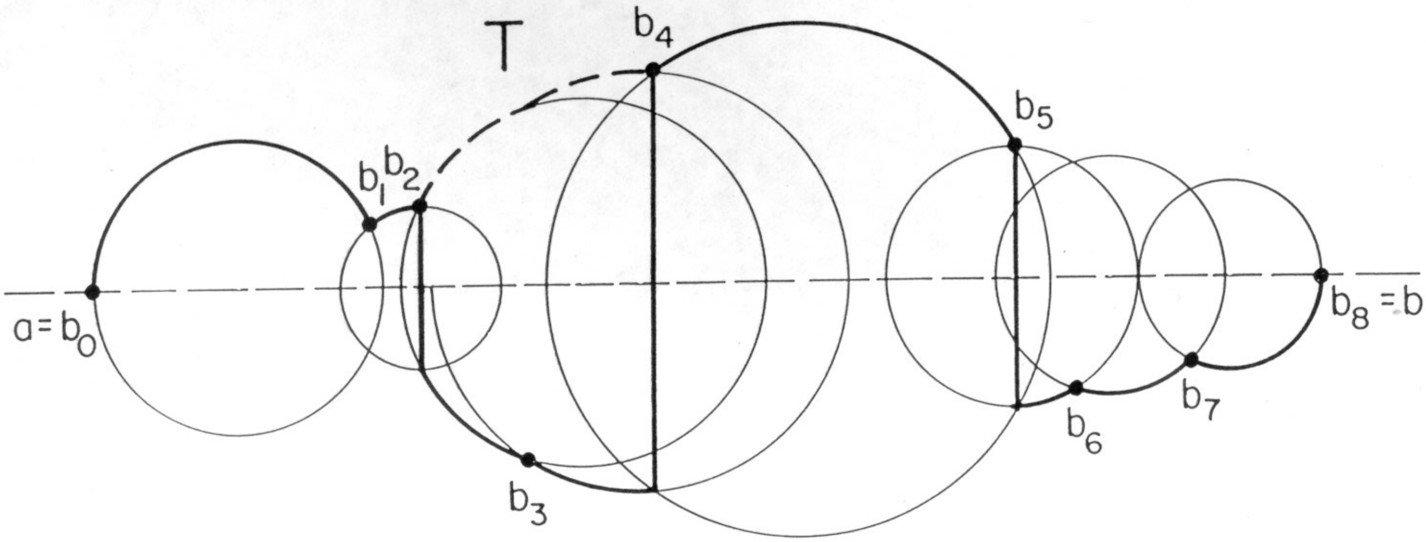

Figure 4

An upper bound on the length of the direct DT path

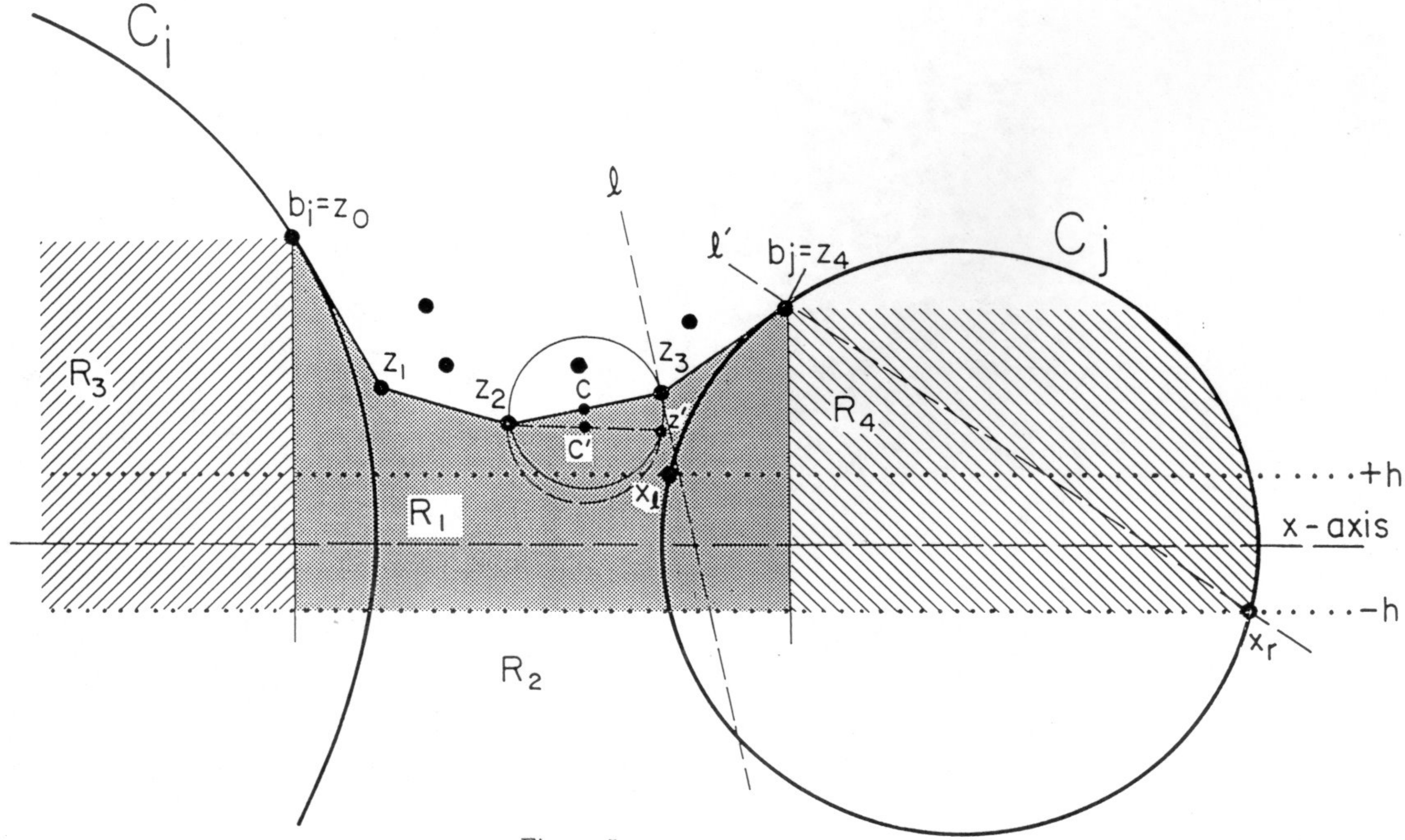

Figure 5

The shortcut from b_i to b_j. Here $k = 2$.

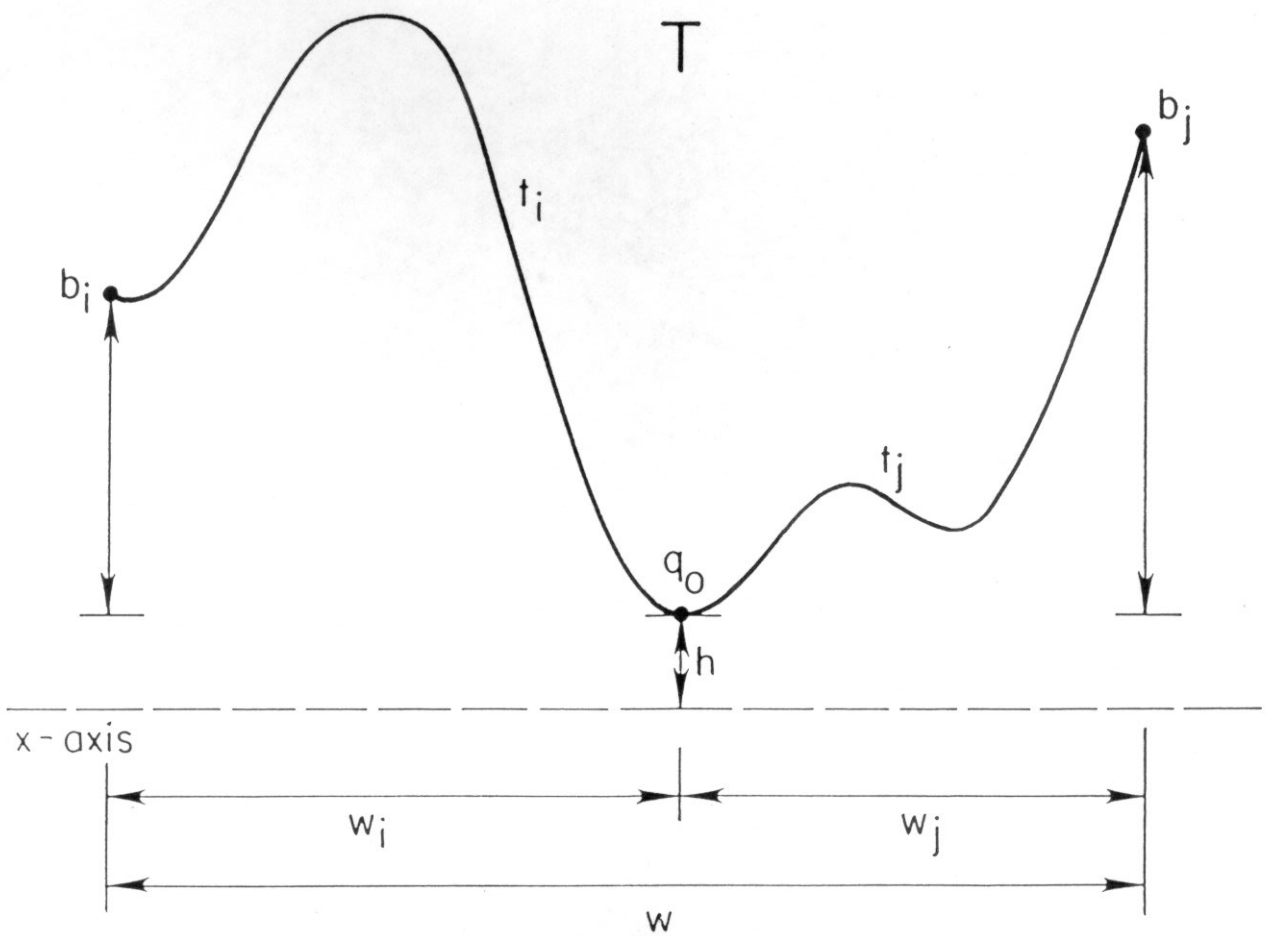

Figure 6

Analyzing the path length when the shortcut is *not* taken

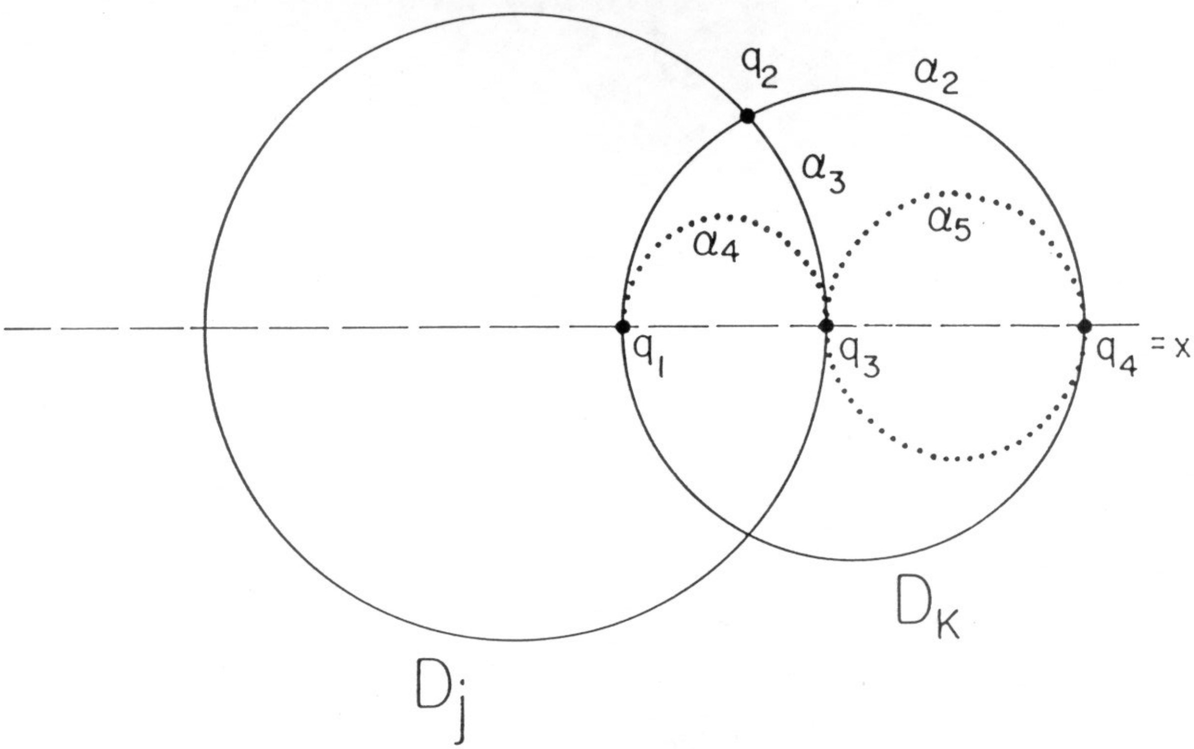

Figure 7

Illustration for the proof of Lemma 3.