PLANARITY TESTING OF DOUBLY PERIODIC INFINITE GRAPHS

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Abstract

This paper describes an efficient way to test the *VAP-free* (Vertex Accumulation Point free) planarity of one- and two-dimensional *dynamic graphs*. *Dynamic graphs* are infinite graphs consisting of an infinite number of basic cells connected regularly according to labels in a finite graph called a *static graph*. Dynamic graphs arize in the design of highly regular VLSI circuits, such as systolic arrays and digital signal processing chips.

We show that VAP-free planarity testing of dynamic graphs can be done efficiently by making use of their regularity. First, we will establish necessary conditions for VAP-free planarity of dynamic graphs. Then we show the existence of a finite graph which is planar if and only if the original dynamic graph is VAP-free planar. From this it follows that VAP-free planarity testing of one- and two-dimensional dynamic graphs is asymptotically no more difficult than planarity testing of finite graphs, and thus can be done in linear time.

1. Introduction

Given a finite digraph $G^0 = (V^0, E^0)$, called a *static graph*, and a k-dimensional labeling of edges T^k, we can define the k-dimensional *dynamic graph* $G^k = (V^k, E^k, T^k)$ as follows: Let $V^0 = \{ v_1, v_2, \ldots, v_n \}$. For each $\mathbf{x} \in Z^k$, we call $v_{i,\mathbf{x}}$ *the* $\mathbf{x}$*-th copy* of $v_i \in V^0$, and $V_\mathbf{x} = \{v_{1,\mathbf{x}}, v_{2,\mathbf{x}}, \ldots, v_{n,\mathbf{x}}\}$ *the* $\mathbf{x}$*-th copy of* V^0. The vertex set $V_\mathbf{x}$ can be regarded as a copy of V^0 at the integer lattice point $\mathbf{x}$ and V^k is the union of all points; that is,

$$V^k = \bigcup_{\mathbf{x} \in Z^k} V_\mathbf{x}.$$

Two vertices $v_\mathbf{x}$ and $w_\mathbf{y}$ in G^k are connected by a copy of an edge (v, w) in G^0 whose label is the same as the distance between these two vertices $(\mathbf{y} - \mathbf{x})$ in k-dimensional space; that is, the edge set E^k is defined as

$$E^k = \{ (v_\mathbf{x}, w_\mathbf{y}) \mid v_\mathbf{x} \in V_\mathbf{x}, w_\mathbf{y} \in V_\mathbf{y}, (v, w) \in E^0, \mathbf{y} - \mathbf{x} = T^k((v, w)) \}.$$

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Hence the dynamic graph is a locally finite infinite graph consisting of an infinite number of repetitions of the basic cell.

Orlin (1984) pointed out that many problems in transportation planning, communications, and operations management can be modeled by one-dimensional dynamic graphs. He investigated various problems for one-dimensional dynamic graphs, such as finding weak or strong components, finding an Eulerian path, and determining whether they are 2-colorable or not.

Two-dimensional dynamic graphs arise naturally in the study of regular VLSI circuits, such as systolic arrays and VLSI signal processing arrays (Cappello and Steiglitz 1981; Iwano and Steiglitz 1986b). In these applications, the graphs associated with the circuits can be regarded as subgraphs of two-dimensional dynamic graphs. Doubly-weighted digraphs, which can be regarded as static graphs of two-dimensional dynamic graphs, have also been well studied. For example, Dantzig, Blatter, and Rao (1967) and Lawler (1967) studied optimal cycles with minimum ratio of two labels; Reiter (1968) studied these graphs for problems of scheduling parallel computation. The authors studied the acyclicity problem (Iwano and Steiglitz 1986a) and various other problems for two-dimensional dynamic graphs (Iwano and Steiglitz 1986c).

The regularity of dynamic graphs may lead us to efficient solutions of certain problems because we may be able to restrict problems to finite graphs which adequately represent them. We will show that *VAP-free* planarity testing of dynamic graphs can be solved efficiently using this idea. The planarity problem of infinite graphs in general has been extensively studied (Dirac and Schuster 1954; Grünbaum and Shephard 1978, 1981; Halin 1966; Thomassen 1977, 1980, 1983). There are efficient planarity testing algorithms for finite graphs (Hopcroft and Tarjan 1974; Lempel, Even, and Lederbaum 1967). An infinite planar graph is *VAP-free* planar if there is no vertex accumulation point in any finite bounded region. In VLSI applications, since each cell occupies at least some constant area, the associated dynamic graph is VAP-free planar if it is planar. Hence we will consider only VAP-free planarity of dynamic graphs.

First, we will find necessary conditions for VAP-free planarity of dynamic graphs. Then we will show the existence of a finite graph which is no larger than a constant multiple times the size of a basic cell and which is planar if and only if the original dynamic graph is VAP-free planar. From this it follows that VAP-free planarity testing can be done in O(n) time where n is the number of vertices in the basic cell.

2. Graph terminology

We will need the following definitions related to the planarity of infinite graphs (Grünbaum and Shephard 1981; Thomassen 1977).

Definition 2.1. A graph $G = (V, E)$ is called a *plane graph* if all vertices and edges lie in a plane π. In this case, the points of the plane not on G are partitioned into open sets called *faces*, or *regions*. A graph G is said to be *planar, have a plane representation,* or *be embeddable in the plane* if it is isomorphic to a plane graph. The plane graph is called a *plane representation of* G. □

Definition 2.2. Given a digraph $G = (V, E)$, a *path* P in G is a sequence of vertices $P = v_0, v_1, \ldots, v_l$ where $e_i = (v_{i-1}, v_i) \in E$ and $v_i \in V$. If all vertices $v_0, v_1, \ldots, v_{l-1}$ are distinct, a path P is *simple*. A path P such that $v_0 = v_l$ is called a *cycle*. □

Definition 2.3. A *countable graph* is one in which both the vertex set and the edge set are finite or countably infinite. A graph is *locally finite* if the valence of every vertex is finite. A point P in an infinite plane graph G is called a *vertex accumulation point* (resp. *edge accumulation point*) if there are infinitely many vertices (resp. edges) of G of Euclidean distance $< \varepsilon$ from P for every positive real number ε. A vertex accumulation point (resp. edge accumulation point) is abbreviated *VAP* (resp. *EAP*). A *two-way infinite path,* abbreviated by $2-\infty$ path, is an infinite sequence of distinct edges of the form

$$\cdots, (v_{-r}, v_{-r+1}), \cdots, (v_{-1}, v_0), (v_0, v_1), \cdots, (v_{r-1}, v_r), \cdots.$$

□

Definition 2.4. A plane graph is *straight* and has a *straight-line representation* if all of its edges are straight line segments. A straight plane graph is *convex* if all of its bounded regions are convex plane sets and its unbounded regions are either convex or complements of convex sets. A plane graph G is said to be a *triangulation* if the boundary of every region is 3-cycle. A plane graph G is said to be a *polygonal arc representation* if the edges of G are polygonal arcs. □

We define a dynamic graph as an infinite graph induced by a finite graph.

Definition 2.5. Let $G^0 = (V^0, E^0)$ be a finite directed graph with $V^0 = \{v_1, v_2, \ldots, v_n\}$. Let $T^k : E^0 \to Z^k$ be a k-dimensional labeling of E^0 such that $T^k(e) = \{e_1, e_2, \ldots, e_k\} \in Z^k$ for every $e \in E^0$. For each $\mathbf{x} \in Z^k$, we call $v_{i,\mathbf{x}}$ *the* $\mathbf{x}$*-th copy* of $v_i \in V^0$, and $V_{\mathbf{x}} = \{v_{1,\mathbf{x}}, v_{2,\mathbf{x}}, \ldots, v_{n,\mathbf{x}}\}$ *the* $\mathbf{x}$*-th copy of* V^0. Then we can define the *k-dimensional dynamic graph* $G^k = (V^k, E^k, T^k)$ induced by G^0 as follows:

$$\begin{cases} V^k = \bigcup_{\mathbf{x} \in Z^k} V_{\mathbf{x}} \\ E^k = \{ (v_{i,\mathbf{x}}, v_{j,\mathbf{y}}) \mid (v_i, v_j) \in E^0, \mathbf{y} - \mathbf{x} = T^k((v_i, v_j)) \}. \end{cases}$$

We call G^0 the *static graph* of G^k. The edge with the $T^k(e)$ label is called the *$T^k(e)$-edge.* □

Note that G^k is an infinite graph and is locally finite. We use $\mathbf{0}$ to represent the origin in Z^k; that is, $\mathbf{0} = (0, 0, \ldots, 0)$. We now define *the basic cell of G^k* as follows:

Definition 2.6. For $\mathbf{x}, \mathbf{y} \in Z^k$, let $E_{\mathbf{x},\mathbf{y}} = \{ (v_{i,\mathbf{x}}, v_{j,\mathbf{y}}) \in E^0 \}$. When $\mathbf{x} \neq \mathbf{y}$, we call $E_{\mathbf{x},\mathbf{y}}$ *the connecting edges.* We call $C_{\mathbf{x}} = (V_{\mathbf{x}}, E_{\mathbf{x},\mathbf{x}})$ *the* $\mathbf{x}$*-th cell of* G^k. In particular, we call $C_{\mathbf{0}}$ the *basic cell of* G^k. When we regard each cell as a point, we have an infinite graph $G_c^k = (V_c^k, E_c^k, T_c^k)$ such that $V_c^k = Z^k$ and $E_c^k = \bigcup_{\mathbf{x} \neq \mathbf{y}} E_{\mathbf{x},\mathbf{y}}$. We call G_c^k *the cell graph of* G^k. □

The graph G_c^k is obtained by regarding every cell of G^k as a point; G^k can be regarded as the union of cells and connecting edges.

Definition 2.7. Let $G_c^k = (V_c^k, E_c^k, T_c^k)$ be the cell graph of a k-dimensional dynamic graph G^k. Then we define the *cell static graph* $G_c^0 = (V_c^0, E_c^0, T_c^k)$ as follows:

$$\begin{cases} V_c^0 = \{ v \} \\ E_c^0 = \{ e = (v, v) \mid e \in E, T(e) \neq \mathbf{0} \} \\ T_c^k = \{ T^k(e) \mid e \in E_c^0 \}. \end{cases}$$

□

This cell static graph G_c^0 is the static graph which induces G_c^k. In Fig. 1a, the two-dimensional dynamic graph G^2 is induced by a static graph G^0, while in Fig. 1b, the cell graph G_c^2 is induced by the cell static graph G_c^0. The cell graph G_c^2 represents the interconnection between cells in the dynamic graph G^2, and the cell static graph G_c^0 consists of edges with non-$\mathbf{0}$ labels in G^0. We use the notations illustrated in Fig. 1c. That is, the superscript k of G indicates a k-dimensional dynamic graph, while the superscript 0 indicates a static graph. The subscript c of G or G^k indicates a cell graph.

From now on, we discuss dynamic graphs G^k with $k \leq 2$.

Definition 2.8. To *subdivide* an edge $e = (x, y)$ in a graph H, is to replace it by a new vertex z, new edges $e_1 = (x, z)$ and $e_2 = (z, y)$. We say that the resulting graph G is obtained from H by *subdividing* e at z. A graph G is a *subdivision* of H if there is a sequence of graphs

$$H_0 = H,\ H_1,\ H_2,\ \ldots,\ H_n = G$$

such that H_i is obtained from H_{i-1} by subdividing an edge in H_{i-1} for $1 \le i \le n$. □

Thomassen (1983) summarized the current results about planarity of infinite graphs. For example, Erdös extended Kuratowski's theorem to countable graphs (Dirac and Shuster 1954) as follows:

Theorem 2.1. A countable graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. □

As another example, Halin characterized locally finite graphs having VAP-free representations:

Theorem 2.2. (Halin 1966) A locally finite graph has a VAP-free representation if and only if it is countable and contains no subdivision of K_5, $K_{3,3}$, or any of the graphs in Fig. 2. □

Fig. 3 shows two representations of a one-dimensional dynamic graph G^1 induced by a static graph G^0 with two connecting edges with labels 2 and 3. Note that Fig. 3a is not a plane graph, while Fig. 3b is a plane graph with a vertex accumulation point. In fact, by using Theorem 2.2, we can show that this dynamic graph does not have a plane representation without a vertex accumulation point. The wide solid lines in Fig. 3c form one of Halin's subgraphs, as shown in Fig. 3d.

Thomassen obtained the following results for straight-line representation and a convex representation.

Theorem 2.3. (Thomassen 1977) Every planar graph has a straight-line representation, and every locally finite graph with a VAP-free representation has a VAP-free straight-line representation. □

Theorem 2.4. (Thomassen 1980) Every locally finite 3-connected graph with a VAP-free representation has a convex representation. □

From now on, we assume every edge in a dynamic or static graph is a *simple curve* (See Berge 1963). We will use Jordan's theorem, which states that a simple closed curve in the plane divides the plane into precisely two regions.

3. Necessary conditions for VAP-free planarity of G^1

In this section, we will express necessary conditions for VAP-free planarity of dynamic graphs in terms of the labels of edges. From now on we assume the following:

1) G^k is connected.

2) The basic cell C_0 is connected and planar.

These can be assumed without loss of generality. Note that G^k is planar if and only if every connected component of G^k is planar. Hence if G^k is not connected, we only have to check the VAP-free planarity of each connected component. Thus we can assume 1). Note that 1) implies that the static graph G^0 is connected, because a non-connected static graph induces a non-connected dynamic graph. If C_0 is not planar, neither is G^k, because C^0 is a subgraph of G^k. Since the static graph is assumed to be connected, we can always choose a 2-dimensional labeling which makes the basic cell C_0 connected and does not change the dynamic graph (Orlin 1984). Thus we can assume 2).

Theorem 3.1. The cell graph G_c^k is planar (resp. VAP-free, convex), if the original dynamic graph G^k is planar (resp. VAP-free, convex).

Proof. Let G^k be a planar (VAP-free, or convex) representation of itself. Then by replacing each cell of G^k by a point, we can get a planar (VAP-free, or convex) representation of G_c^k. $\square$

Thomassen showed the following about VAP-free, locally finite plane graphs.

Theorem 3.2. (Thomassen 1977) Let G be an infinite, locally finite, connected VAP-free plane graph. Then there exists an infinite straight line triangulation Δ of the plane such that G is isomorphic to a subgraph of Δ. $\square$

Note that dynamic graphs are locally finite by definition. Thus we can apply Theorem 3.2 to any connected VAP-free plane dynamic graph and show that its vertex set can be chosen to be integer lattice points of the plane as follows:

Corollary 3.1. Let G^2 be a connected, VAP-free, plane graph. Then G^2 is isomorphic to a subgraph of a plane graph $\Gamma = (\Gamma_V, \Gamma_E)$ where $\Gamma_V \subset Z^2$.

Proof. Let Δ be an infinite straight line triangulation of the plane such that G is isomorphic to a subgraph of Δ. Let $p_0p_1p_2$ be a triangle of Δ. If necessary, we can expand the triangle $p_0p_1p_2$ so that it contains at least three integer points. Let $q_0q_1q_2$ be a triangle such that q_0, q_1, and q_2 are integer points in the triangle $p_0p_1p_2$. We can then replace the triangle $p_0p_1p_2$ by the triangle $q_0q_1q_2$. By repeating this operation, we can obtain a triangulation of the plane Δ' whose vertices are integer points. Thus G is isomorphic to a subgraph of Δ'. $\square$

Let $G_c^1 = (V_c^1, E_c^1, T_c^1)$ be the cell graph of a one-dimensional dynamic graph G^1 and let $G_c^0 = (V_c^0, E_c^0, T_c^1)$ be the cell static graph with

$$\begin{cases} V_c^0 = \{ v \} \\ E_c^0 = \{ e_1, e_2, \ldots, e_m \} \text{ where} \\ \quad e_i = (v, v) \text{ and } T_c^1(e_i) = x_i \text{ such that} \\ \quad 0 < | x_1 | \leq | x_2 | \leq \cdots \leq | x_m |. \end{cases} \tag{3.1}$$

Since we are concerned with planarity, without loss of generality, we can assume that $x_i > 0$ for $1 \leq i \leq m$, and that the edge-labels of G_c^0 are distinct, so that

$$0 < x_1 < x_2 < \cdots < x_m. \tag{3.2}$$

We have the following definition about $2-\infty$ paths induced by a p-edge (that is, an edge with label p).

Definition 3.1. Let each vertex of V_c^1 be denoted by an integer. Suppose that there is a p-edge in G_c^1. Then each p-edge in G_c^1 induces a $2-\infty$ path $P_{p,i} = (V_{p,i}, E_{p,i})$ for $0 \leq i \leq p - 1$ as follows:

$$\begin{cases} V_{p,i} = \{ n \mid n \equiv i \ (mod\ p) \}, \\ E_{p,i} = \{ (n, n + p) \mid n \in V_{p,i} \}. \end{cases}$$

□

That is, $P_{p,i}$ is a $2-\infty$ path consisting of p-edges and the nodes which are equal to $i \ mod\ p$. Note that V_c^1 is the disjoint union of $\{ V_{p,i} \mid 0 \leq i \leq p - 1 \}$.

From Theorem 3.1, VAP-free planarity of the cell graph G_c^k is a necessary condition for VAP-free planarity of dynamic graph G^k. Therefore, we have the following necessary conditions for VAP-free planarity of one-dimensional dynamic graphs:

Theorem 3.3. Let G^1 be a connected one-dimensional dynamic graph. Let G_c^0 be the cell static graph as defined in (3.1) and (3.2). Then G_c^1 is VAP-free planar if and only if one of the following two conditions is satisfied. (See Fig. 4.)

1) $m = 1$ and $x_1 = 1$.
2) $m = 2$, $x_1 = 1$ and $x_2 = 2$.

Before proving Theorem 3.3, we need the following lemmas:

Lemma 3.2. (Thomassen 1980) Let G be a VAP-free and EAP-free representation of a $2-\infty$ path. Then G partitions the Euclidean plane precisely into two faces. □

Lemma 3.3. Let G be a locally finite VAP-free plane graph. Then G is EAP-free.

Proof. Suppose that G is not EAP-free. Then there exists a bounded closed area containing infinitely many edges. However, since G is locally finite (that is, every vertex has a finite valence), there are infinitely many vertices in this closed area, which is a contradiction. □

Now we can prove Theorem 3.3.

Proof of Theorem 3.3. The "if" part is easy. As shown in Fig. 4, both cases have VAP-free planar representations.

We can now prove the "only if" part. Suppose that $G_c^1 = (V_c^1, E_c^1, T_c^1)$ is a VAP-free planar representation. From Corollary 3.1, we can assume that the vertex set V_c^1 consists of integer lattice points in $Z \times Z$.

Suppose that $x_1 \geq 2$. Since G_c^1 is connected, there exists some x_j such that $j \geq 2$ and x_j is not a multiple of x_1. Otherwise, node 0 and node 1 cannot be connected, which is a contradiction. Let x_1 (resp. x_j) be denoted by p (resp. q). Then there exist some k, $r \in Z^+$ such that $q = kp + r$, $0 < r < p$. From Lemma 3.2, the set of $2-\infty$ paths $\{P_{q,j}\}$ partitions the Euclidean plane into $(q+1)$ faces. Note that the $2-\infty$ path $P_{p,0}$ connects nodes

$$0 \rightarrow p \rightarrow 2p \rightarrow \cdots \rightarrow (q-1)p \rightarrow qp$$

such that $ip \in P_{q,ip(mod\ q)}$ for $0 \leq i \leq q$. Therefore, the $(q+1)$ faces created by $\{P_{q,j}\}$ are arranged in the following order:

$$P_{q,0}, P_{q,p}, \cdots, P_{q,kp}, P_{q,(k+1)p} = P_{q,(p-r)}, \cdots$$

as shown in Fig. 5. Note that 0, p, and $2p$ are different from each other *mod q*, and thus $2-\infty$ paths $P_{q,0}$, $P_{q,p}$, and $P_{q,2p}$ are different from each other. Now we have the following two closed undirected cycles W_1 and W_2 in G_c^1 as illustrated by the wide solid lines in Fig. 5:

$$W_1 \colon 0 \rightarrow p \rightarrow 2p \rightarrow \cdots \rightarrow qp \rightarrow 0$$

and

$$W_2 \colon 0 \rightarrow -q \rightarrow (p-q) \rightarrow (2p-q) \rightarrow 2p \rightarrow p \rightarrow 0.$$

Note that W_2 uses $P_{p,-q}$, $P_{q,2p}$, $P_{p,0}$, and $P_{q,0}$. Note also that $P_{p,0}$ connects $2p \in P_{q,2p}$ and $qp \in P_{q,0}$ through $(q-1)p \in P_{q,(q+1)p}$. Let

$$P_{q,p}^{+} = \{ p + np \mid n \in Z^+ \} \subset P_{q,p}.$$

Since there is no vertex on W_1 and W_2 which is also a vertex in $P_{q,p}^{+}$, the $1-\infty$ path $P_{q,p}^{+}$ must cross W_1 and W_2 to get from the inside of those cycles to the outside. Therefore, $P_{q,p}^{+}$ must remain inside W_1 and W_2 in order to remain planar. If $p+q$ is inside W_1, then $P_{q,p}^{+}$ is also inside W_1. This implies a VAP in W_1, which is a contradiction. In the same way, $p+q$ cannot be inside W_2. Therefore, $x_1 = 1$.

Suppose that $x_2 > 2$. Since $x_1 = 1$, from Lemma 3.2, the $2-\infty$ path $P_{1,0}$ partitions the plane into precisely two faces, say the upper face and the lower face. Suppose $P_{x_2,0}$ exists in the upper face, then $P_{x_2,1}$ should exist in the lower face, as shown in Fig. 6. Note that node 2 is located in the closed region

$$C_1 : 0 \to 1 \to (x_2+1) \to x_2 \to 0,$$

while node (x_2+2) is located in the closed region

$$C_2 : x_2 \to (x_2+1) \to (2x_2+1) \to 2x_2 \to x_2 .$$

Thus there is no way to connect node 2 and node (x_2+2) without crossing $P_{x_2,1}$ or $P_{x_2,0}$, which is a contradiction. Therefore, if $m \geq 2$, m should be 2 and $x_1 = 1$, $x_2 = 2$. □

4. VAP-free planarity testing of G^1

In this section we will show that VAP-free planarity testing of one-dimensional dynamic graphs can be done in O(n) time where n is the number of vertices in the basic cell. We use a finite graph G_f instead of the infinite graph G^1 to test VAP-free planarity of G^1. The graph G_f associated with G^1 is defined as follows:

Definition 4.1. Let $G^1 = (V^1, E^1, T^1)$ be a one-dimensional dynamic graph. Let $C_x = (V_x, E_{x,x})$ be the x-th cell of G^1 for $x \in Z$ where $E_{x,y}$ is the set of connecting edges between the x-th and the y-th cell as in Definition 2.6. Then we can define the finite graph $G_f = (V_f, E_f)$ as follows:

$$\begin{cases} V_f = V_0 \cup V_1 \cup V_2 \cup V_3 \cup \{ s, t \}, \\ E_f = \{ E_{x,y} \mid 0 \leq i \leq j \leq 3 \} \cup \\ \qquad \{ (s, w) \mid \textit{there exists some } v \textit{ s.t. } (v, w) \in E_{x,y},\ x < 0 \leq y \leq 3 \} \cup \\ \qquad \{ (v, t) \mid \textit{there exists some } w \textit{ s.t. } (v, w) \in E_{x,y},\ 0 \leq x \leq 3 < y \} \cup \\ \qquad \{ (s, t) \}. \end{cases}$$

□

Fig. 7 shows an example of G_f. Note that the vertex s (resp. t) represents the cells of G^1 for $i < 0$ (resp. $i > 3$).

From Theorem 3.3, we can assume the following:

1) The cell graph of G^1 satisfies $E_{i,j} = \varnothing$ for $\mid i - j \mid \geq 3$ and $E_{i,i+1} \neq \varnothing$ for $i \in Z$ (that is, there is a 1-edge and no p-edge for $p > 2$).
2) The basic cell is connected and planar.

Then we have the following theorem:

Theorem 4.1. A one-dimensional dynamic graph G^1, which satisfies the above assumptions, has a VAP-free planar representation if and only if the associated finite graph G_f is planar.

Proof. Suppose that G_f is planar. Assume there is a 2-edge. (If not, the following proof can be easily modified.) Since there is at least a 1-edge and since the basic cell is connected, there is an undirected cycle

$$W: s \to C_0 \to C_1 \to C_2 \to C_3 \to t \to s$$

in G_f. Without loss of generality, we can assume that s, C_0, C_1, C_2, C_3, and t are located in this order from the left as shown in Fig. 7. Otherwise we can transform the graph to the desired form, without losing VAP-free planarity, by expanding the edge (s, t) and rotating the graph along with the cycle W. From Theorem 2.5 (Jordan's theorem), the cycle W partitions the plane into exactly two regions. We call the inside (resp. outside) R_{in} (resp. R_{out}). Note that the cycle W corresponds to the $2-\infty$ path $P_{1,0}$ in G. Note also that all edges in $E_{0,2}$ lie in either R_{in} or R_{out}, and the same is true for $E_{1,3}$. If $E_{0,2}$ lies in R_{in} (resp. R_{out}), $E_{1,3}$ should lie in R_{out} (resp. R_{in}). Let B be a closed region which contains only C_1 and C_2, as shown by the shaded area in Fig. 7. Then we can obtain a VAP-free representation of G^1 by infinitely repeating B, because we can maintain the same sequence of 2-edges on the boundary of B.

Conversely, suppose that G^1 is VAP-free planar. We can assume that G^1 itself is a VAP-free plane graph. It is clear that the subgraph consisting of C_{-1}, C_0, C_1, C_2, C_3, and C_4 is planar. Then G_f is obtained by contracting C_{-1} (resp. C_4) to the point s (resp. t) and adding the edge (s, t). □

Corollary 4.1. VAP-free planarity testing can be done in O(n) time for a one-dimensional dynamic graph G^1 where n is the number of vertices in the basic cell of G^1.

Proof. We can use any planarity testing algorithm which runs in time linear in the order of the vertex set (Hopcroft and Tarjan 1979; Lempel, Even, and Cederbaum 1967). □

5. Necessary conditions for VAP-free planarity of G^2

We also have similar necessary conditions for VAP-free planarity of two-dimensional dynamic graphs. Let $G_c^0 = (V_c^0, E_c^0, T_c^2)$ be the cell static graph with

$$\begin{cases} V_c^0 = \{ v \} \\ E_c^0 = \{ e_1, e_2, \ldots, e_m \} \\ T_c^2(e_i) = \mathbf{e}_i = (x_i, y_i) \quad \text{for } 1 \le i \le m. \end{cases}$$

As in Section 3, we can assume that $x_i > 0$ for $1 \leq i \leq m$ and $\mathbf{e}_i \neq \mathbf{e}_j$ for $i \neq j$. Let $G_c^2 = (V_c^2, E_c^2, T_c^2)$ be the cell graph of G^2 with

$$\begin{cases} V_c^2 = Z \times Z \\ E_c^2 = \bigcup_{\mathbf{x},\mathbf{y} \in Z \times Z,\ \mathbf{x} \neq \mathbf{y}} E_{\mathbf{x},\mathbf{y}} \quad \text{where} \\ \qquad E_{\mathbf{x},\mathbf{y}} = \{ e_{\mathbf{x},\mathbf{y}} \mid e \in E_c^0,\ T_c^2(e) = \mathbf{y} - \mathbf{x} \}. \end{cases}$$

Theorem 5.1. The cell graph G_c^2 is VAP-free planar if and only if one of the following two conditions is satisfied:

1) $m = 2$ and $| x_1 y_2 - x_2 y_1 | = 1$; that is, every point $p \in Z \times Z$ can be expressed in the form of $a\mathbf{e}_1 + b\mathbf{e}_2$ for some $a,\ b \in Z$.

2) $m = 3$, $| x_1 y_2 - x_2 y_1 | = 1$ and $\mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_1$, or $\mathbf{e}_1 + \mathbf{e}_2$; that is, $\mathbf{e}_3$ is a diagonal line of the parallelogram (0, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2$).

Before proving Theorem 5.1, we need the following lemma:

Lemma 5.1. Let W be a cycle in G_c^2 such that

$$W: p_0 \rightarrow p_1 \rightarrow \cdots \rightarrow p_m \rightarrow p_0$$

for $p_i \in V_c^2$. Suppose there exists a point $q \in V_c^2$ inside W and some $e \in E_c^0$ such that $q + n\mathbf{e} \neq p_j$ for any p_j on W and for any $n \in Z$. Then if G_c^2 is planar, there exists a vertex-accumulation point inside W.

Proof. Note that q and $q + \mathbf{e}$ are connected by an edge $e_{q,q+\mathbf{e}}$ of length $\mathbf{e}$. Since $q + \mathbf{e}$ is not on W, $q + \mathbf{e}$ is either outside or inside W. If $q + \mathbf{e}$ is outside W, $e_{q,q+\mathbf{e}}$ must cross W, a contradiction to the planarity of G_c^2. Hence $q + \mathbf{e}$ is inside W. For the same reason, $\{ q + n\mathbf{e} \mid n \in Z \}$ must be contained inside W. This implies the existence of a vertex-accumulation point in W. $\square$

Lemma 5.2. Let $\mathbf{e}_i = (x_i, y_i) \in Z \times Z$ for $i = 1, 2$. Every point $p \in Z \times Z$ can be expressed in the form $a\mathbf{e}_1 + b\mathbf{e}_2$ for some $a,\ b \in Z$ if and only if $| x_2 y_1 - x_1 y_2 | = 1$.

Proof. The matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ is non-singular if and only if there are some integers a, b, c, and d such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}.$$

$\square$

Now we prove Theorem 5.1.

Proof of Theorem 5.1. The "if" part is trivial. The "only if" part is as follows: If $m = 1$, G_c^2 cannot be connected. Therefore, $m \geq 2$. Suppose that there are no edges $\mathbf{e}_1, \mathbf{e}_2 \in E_c^0$ such that $| x_2 y_1 - x_1 y_2 | = 1$. From Lemma 5.2, there is a point p which cannot be expressed in the form of $a\mathbf{e}_1 + b\mathbf{e}_2$ with $a, b \in Z$. Note that the plane is partitioned by disjoint rectangles $\{ R_{a,b} \mid a, b \in Z \}$ where $R_{a,b}$ is the rectangle whose vertices are $a\mathbf{e}_1 + b\mathbf{e}_2$, $(a+1)\mathbf{e}_1 + b\mathbf{e}_2$, $(a+1)\mathbf{e}_1 + (b+1)\mathbf{e}_2$, and $a\mathbf{e}_1 + (b+1)\mathbf{e}_2$. Since p is in the plane, there exists a rectangle $R_{a,b}$ which contains p. Note that for any $n \in Z$, $p + n\mathbf{e}_1$ cannot be expressed in the form $a\mathbf{e}_1 + b\mathbf{e}_2$ with $a, b \in Z$. Therefore, from Lemma 5.1, there is a vertex-accumulation point in $R_{a,b}$, which is a contradiction. Thus there are two edges $\mathbf{e}_1, \mathbf{e}_2 \in E_c^0$ such that $| x_2 y_1 - x_1 y_2 | = 1$. Now every integer lattice point in the plane is a vertex in some rectangle $R_{a,b}$. If $m \geq 3$, a diagonal line of each rectangle $R_{a,b}$ is the only possible edge which keeps VAP-free planarity. $\square$

6. VAP-free planarity testing of G^2

In this section we will show that VAP-free planarity testing of two-dimensional dynamic graphs can be done in O(n) time where n is the number of vertices in the basic cell. We use the same technique as the one used for VAP-free planarity testing of G^1 in Section 4. That is, we can define the finite graph G_f associated with the infinite graph G^2 and show that G_f is planar if and only if G^2 is VAP-free planar.

Without loss of generality, we can assume the following:

1) $m = 2, 3$ and $\mathbf{e}_1 = (0, 1)$, $\mathbf{e}_2 = (1, 0)$, and $\mathbf{e}_3 = (1, 1)$ if $m = 3$.
2) The basic cell is connected and planar.

The graph G_f associated with G^2 is defined as follows:

Definition 6.1. Let $G^2 = (V^2, E^2)$ be a two-dimensional dynamic graph. Let $C_{\mathbf{x}} = (V_{\mathbf{x}}, E_{\mathbf{x},\mathbf{x}})$ be the $\mathbf{x}$-th cell of G^2 for $\mathbf{x} \in Z \times Z$. Then we can define $G_f = (V_f, E_f)$ as follows:

$$\begin{cases} V_f = \{ v_{\mathbf{x}} \mid \mathbf{x} \in [-1, 1] \times [-1, 1] \} \\ E_f = \{ E_{\mathbf{x},\mathbf{y}} \mid \mathbf{x}, \mathbf{y} \in [-1, 1] \times [-1, 1] \}. \end{cases}$$

$\square$

Theorem 6.1. A two-dimensional dynamic graph G^2, which satisfies the conditions above, is VAP-free planar if and only if the associated finite graph G_f is planar.

Proof. Suppose that G^2 is planar. Since G_f is a finite subgraph of G^2, G_f is also planar.

Conversely, suppose that G_f is planar. Since every cell is connected, there is a cycle W connecting $C_{-1,-1}$, $C_{1,-1}$, $C_{1,1}$, and $C_{-1,1}$. We can assume that $C_{0,0}$ is located inside the cycle W. Let B be a rectangle which contains only $C_{0,0}$ as shown in Fig. 8. Then a VAP-free representation of G^2 is obtained by repeating B at each cell. □

Corollary 6.1 The VAP-free planarity testing can be done in O(n) time for the connected two-dimensional dynamic graph G^2 where n is the number of vertices in the basic cell of G^2.

Proof. The planarity testing can be done in O($| V_f |$) time (Hopcroft and Tarjan 1979; Lempel, Even, and Cederbaum 1967) and $| V_f | =$ O(n). □

7. Conclusions

We investigated VAP-free planarity testing of one- and two-dimensional dynamic graphs. First, we showed necessary conditions for VAP-free planarity of dynamic graphs in terms of the edge labels. Then we showed that there is a finite graph which is no larger then a constant multiple times the size of the basic cell and is planar if and only if the original dynamic graph is VAP-free planar. Therefore, VAP-free planarity testing of dynamic graphs can be done in O(n) time where n is the number of vertices in the basic cell.

Generally speaking, the regularity of dynamic graphs makes problems like planarity-testing easier, because we can transform them to problems of static graphs or sufficiently small finite graphs. Using this idea, the authors are now investigating other problems for two-dimensional dynamic graphs, such as weak connectivity, strong connectivity, Eulerian paths, 2-colorability, the graph thickness problem, and the longest path problem. (Iwano and Steiglitz 1986c).

References

1. Berge, C. 1963. *Topological Spaces*, translated by E. M. Patterson, The Macmillan Company, New York.
2. Cappello, P. R. and K. Steiglitz 1981. "Digital Signal Processing Applications of Systolic Algorithms," *CMU Conference on VLSI Systems and Computations*, H. T. Kung, Bob Sproull, and Guy Steele (eds.), Computer Science Press, Rockville, Md.
3. Christofides, N. 1975. *Graph Theory: An Algorithmic Approach*, Academic Press, London.
4. Dantzig, G. B., W. O. Blattner, and M. R. Rao 1967. "Finding a cycle in a graph with minimum cost to time ratio with application to a ship routing problem," in *Int. Symp. on Theory of Graphs*, 77-83, see Rosentiehl 1967.

5. Dirac, G. A. and S. Schuster 1954. "A theorem of Kuratowski," *Indag. Math.* 16, 343-48.

6. Grünbaum, B. and G. C. Shephard 1978. "Isohedral tilings of the plane by polygons," *Comment. Math. Helv.* 53, 542-71.

7. ———. 1981. "The geometry of planar graphs," *Combinatorics* edited by N. Y. Temperley, London Math. Soc. Lecture Notes 52, Cambridge Univ. Press, London, 124-50.

8. Halin, R. 1966. "Zur häufungspunktfreien Darstellung abzählbarer Graphen in der Ebene," *Arch. Math.* (Basel) 17, 239-43.

9. Hopcroft, J. and R. E. Tarjan 1974. "Efficient Planarity Testing," *JACM*, vol. 21, No. 4, October, 549-68.

10. Iwano K. and K. Steiglitz 1986a. "A semiring on convex polygons and zero-sum cycle problems," Tech. Rep. CS-TR-053-86, Computer Science Dept., Princeton Univ., Princeton, NJ.

11. ———. 1986b. "Optimization of one-bit full adders embedded in regular structures," *IEEE Transaction on Acoustics, Speech, and Signal Processing*, October.

12. ———. 1986c. "Two-dimensional dynamic graphs and their VLSI applications," in preparation.

13. Lempel, A., S. Even, and I. Cederbaum 1967. "An Algorithm for Planarity Testing of Graphs," in *Int. Symp. on Theory of Graphs*, 215-32, see Rosentiehl 1967.

14. Lawler, E. L. 1967. "Optimal cycles in doubly weighted directed linear graphs," in *Int. Symp. on Theory of Graphs*, 209-13, see Rosentiehl 1967.

15. Orlin, J. 1984. "Some problems on dynamic/periodic graphs," in *Progress in Combinatorial Optimization*, 273-93, see Pulleyblank 1984.

16. Pulleyblank, W. R. ed. 1984. *Progress in Combinatorial Optimization*, Academic Press, Orlando, Florida.

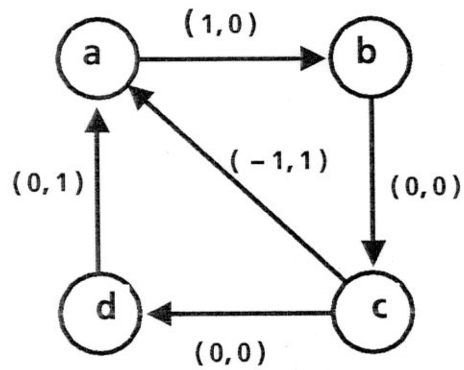
17. Reiter, R. 1968. "Scheduling parallel computation," *Journal of ACM*, 15, No. 4, October, 590-99.

18. Rosentiehl, P. ed. 1967. *Int. Symp. on Theory of Graphs*, Dunod, Paris, Gordon and Breach, New York.

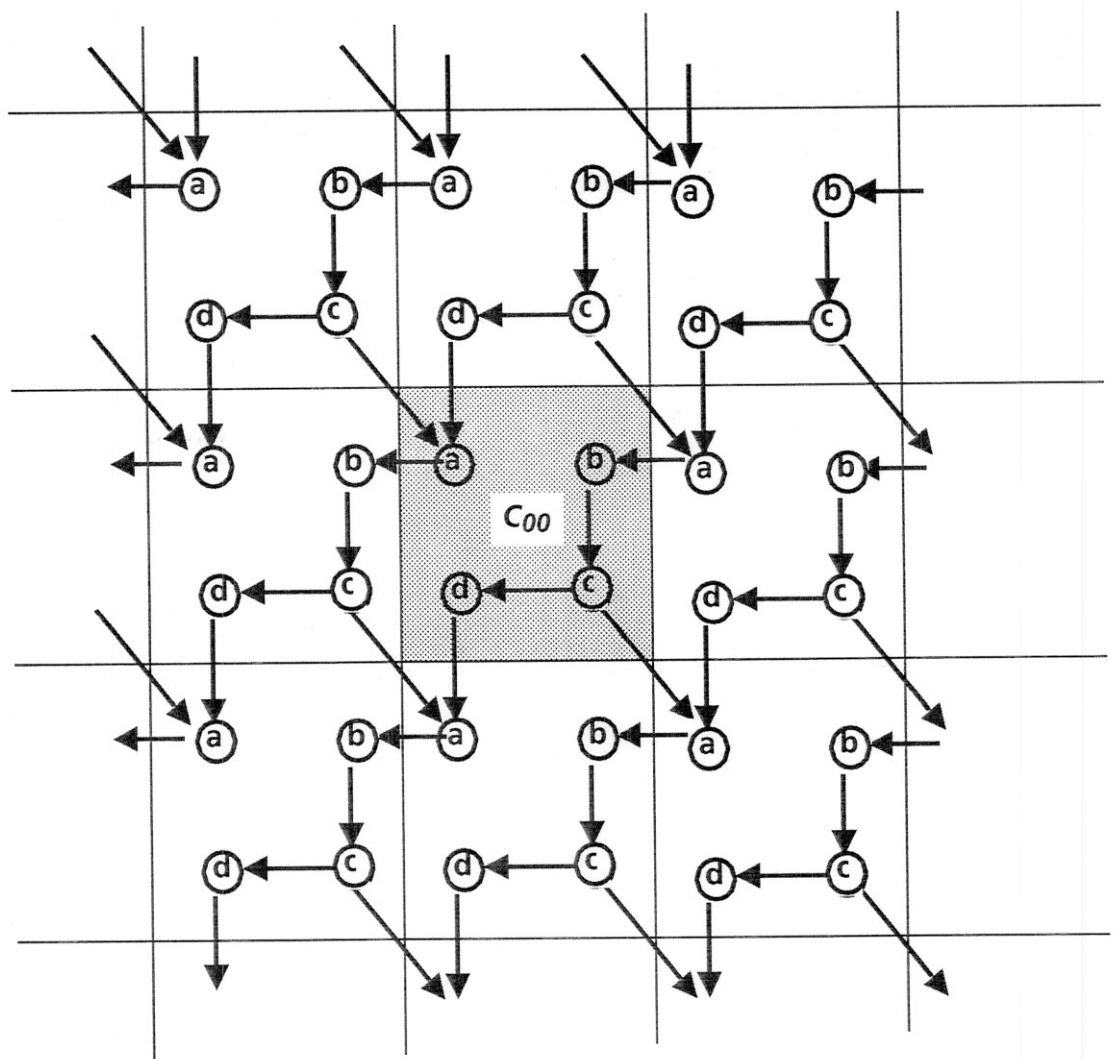
19. Thomassen, C. 1977. "Straight line representations of infinite planar graphs," *J. London Math. Soc.* (2) 16, 411-23.

20. ———. 1980. "Planarity and duality of finite and infinite graphs," *J. Combinatorial Theory (B)* 29, 244-71.

21. ———. 1983. "Infinite Graphs," in *Selected Topics in Graph Theory 2* edited by L. W. Beineke and R. J. Wilson, Academic Press, New York, NY.

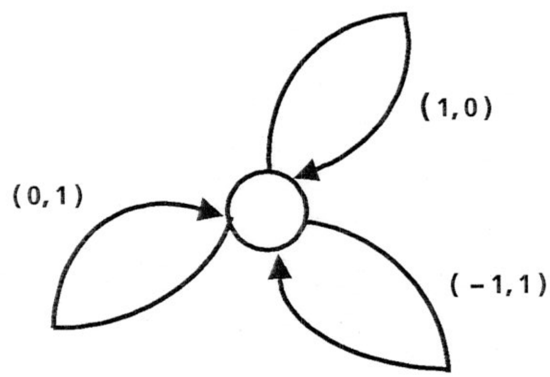


A static graph G^0

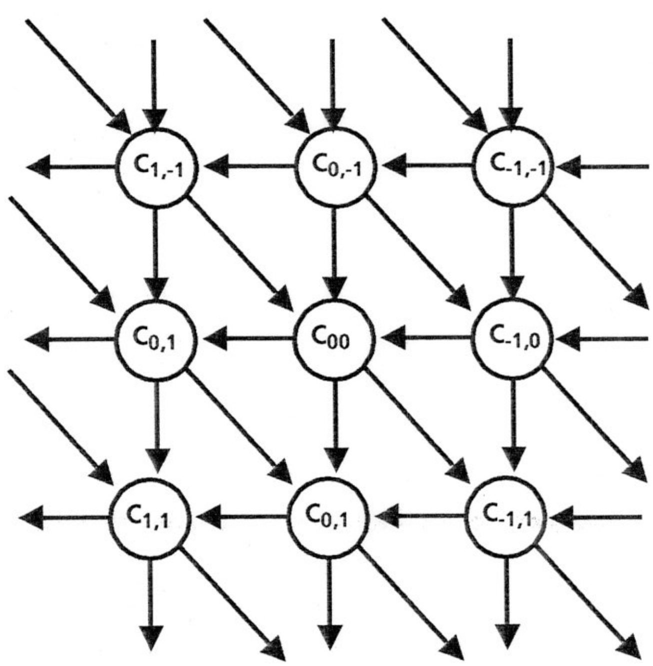


The dynamic graph G^2

Figure 1a. A static graph G shows how to connect the nodes in G^2. The shaded area shows the basic cell C_{00}.



The cell static graph G_c^0



The cell graph G_c^2

Figure 1b. The cell graph G_c^2 indicates the interconnection of cells in the dynamic graph G^2.

$$\begin{array}{ccc} G^0 & \rule[0.5ex]{10em}{0.4pt} & G^k \\ \text{(a static graph)} & & \text{(the } k\text{-dim. dynamic graph)} \\ \Big| & & \Big| \\ G_c^0 & \rule[0.5ex]{10em}{0.4pt} & G_c^k \\ \text{(the cell static graph)} & & \text{(the } k\text{-dim. cell graph)} \end{array}$$

Figure 1c. The superscript *0* indicates a static graph, while the superscript *k* indicates a *k*-dimensional dynamic graph. The subscript *c* indicates a cell graph.

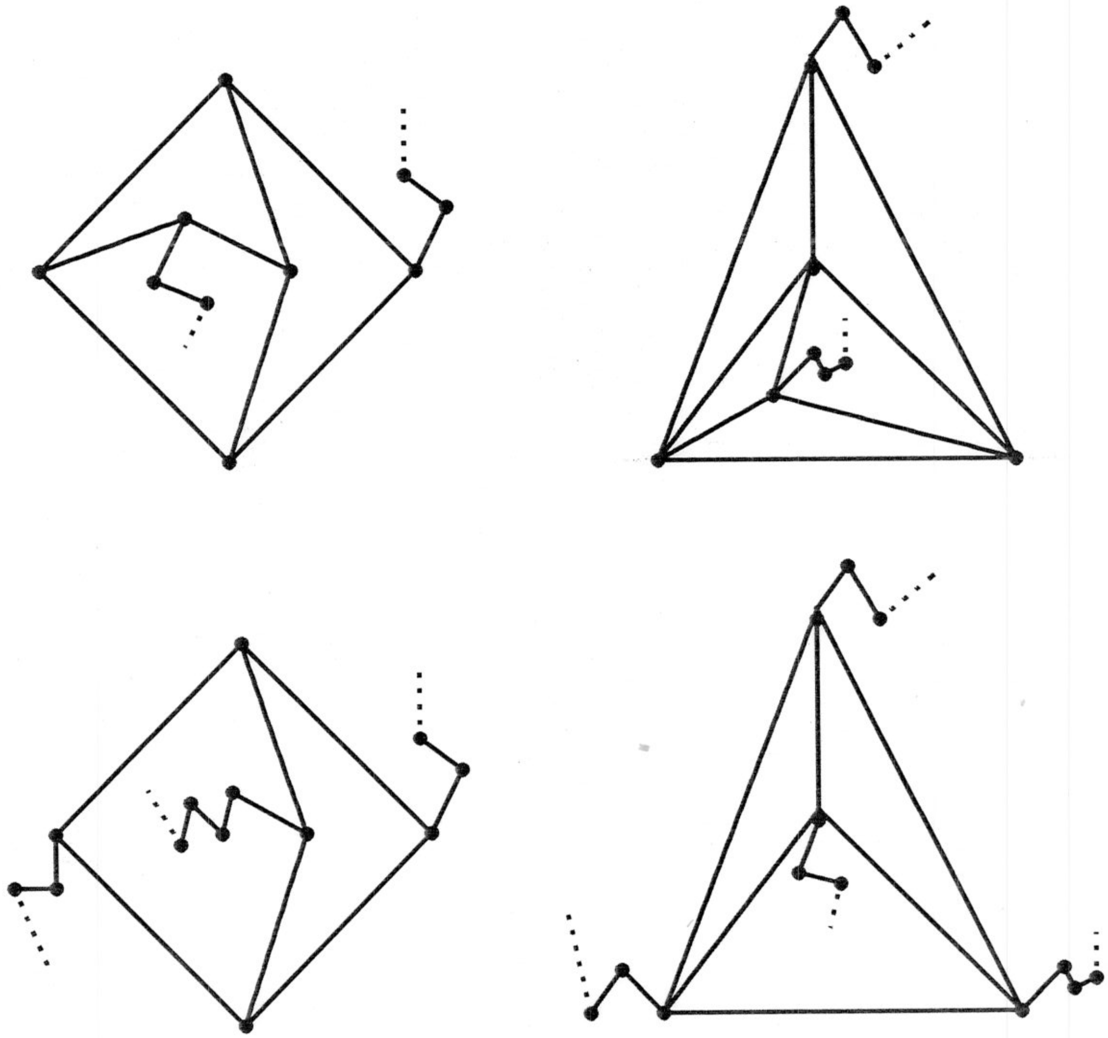
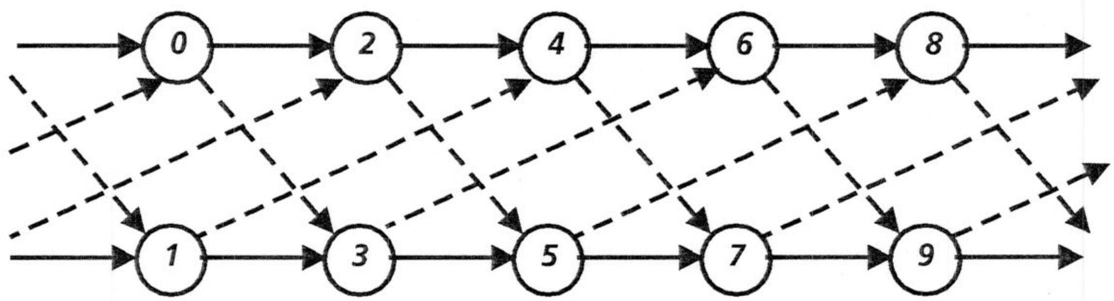
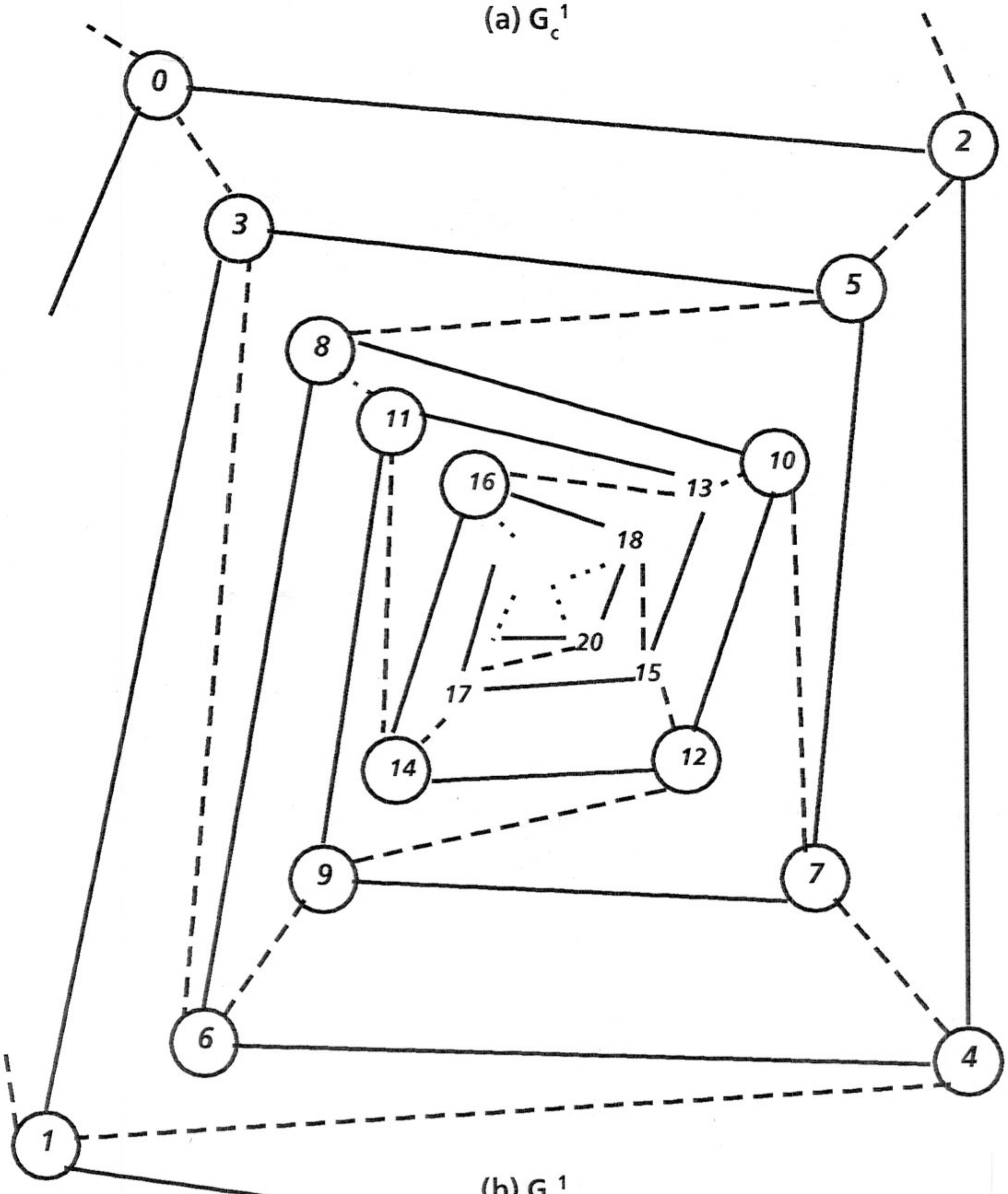


Figure 2. A locally finite graph has a VAP-free representation if and only if it is countable and contains no subdivision of K_5, $K_{3,3}$, or any of the above graphs. The dotted lines denote one-way infinite paths.

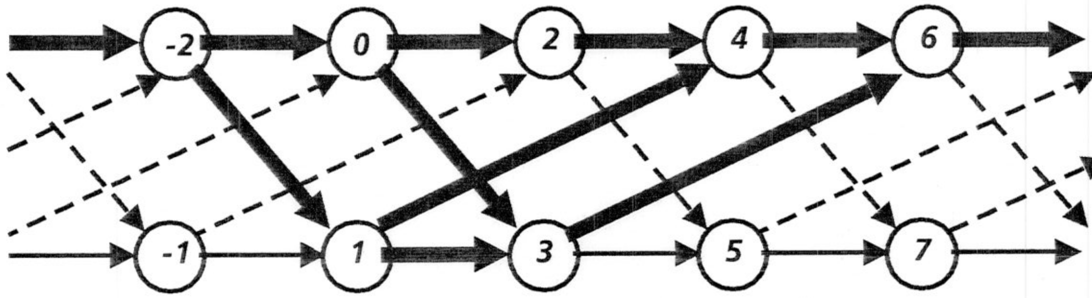


(a) G_c^1

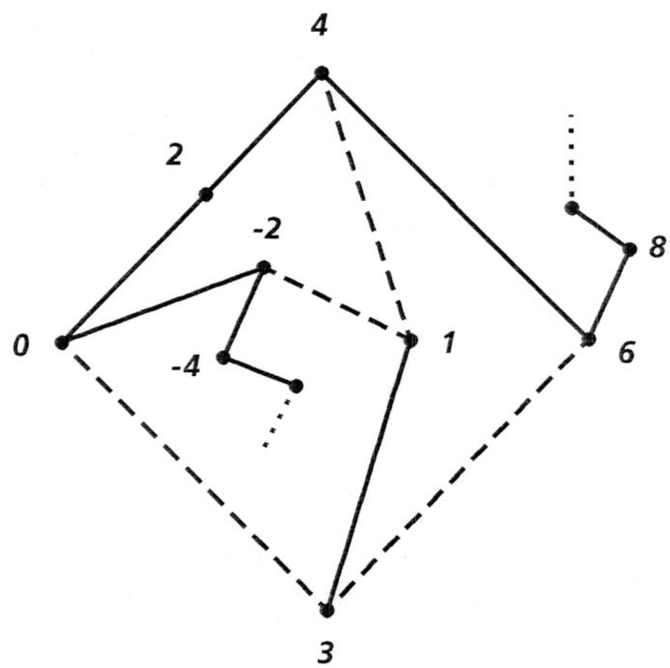


(b) G_c^1

Figure 3. These are representations of the one dimensional dynamic graph with $x_1 = 2$ and $x_2 = 3$. Note that (a) is a non-plane graph and (b) is a plane graph with a VAP.

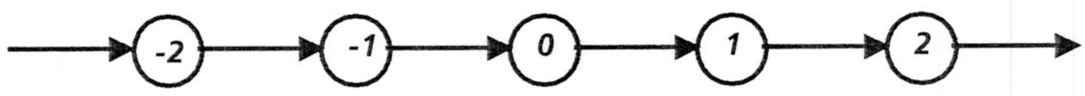


(c) G_c^1

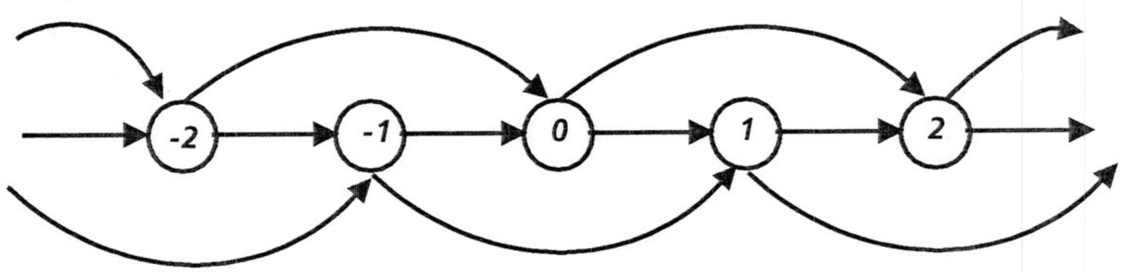


(d)

Figure 3. (c) has a subgraph corresponding to one of Halin's graphs as shown in (d). Therefore $G_c^{\,1}$ cannot have a VAP-free planar representation.



$m = 1, x_1 = 1$



$m = 2, x_1 = 1, x_2 = 2$

Figure 4. The two cases above are the VAP-free planar representations of G_c^{1} with $|x_i| \leq 2$.

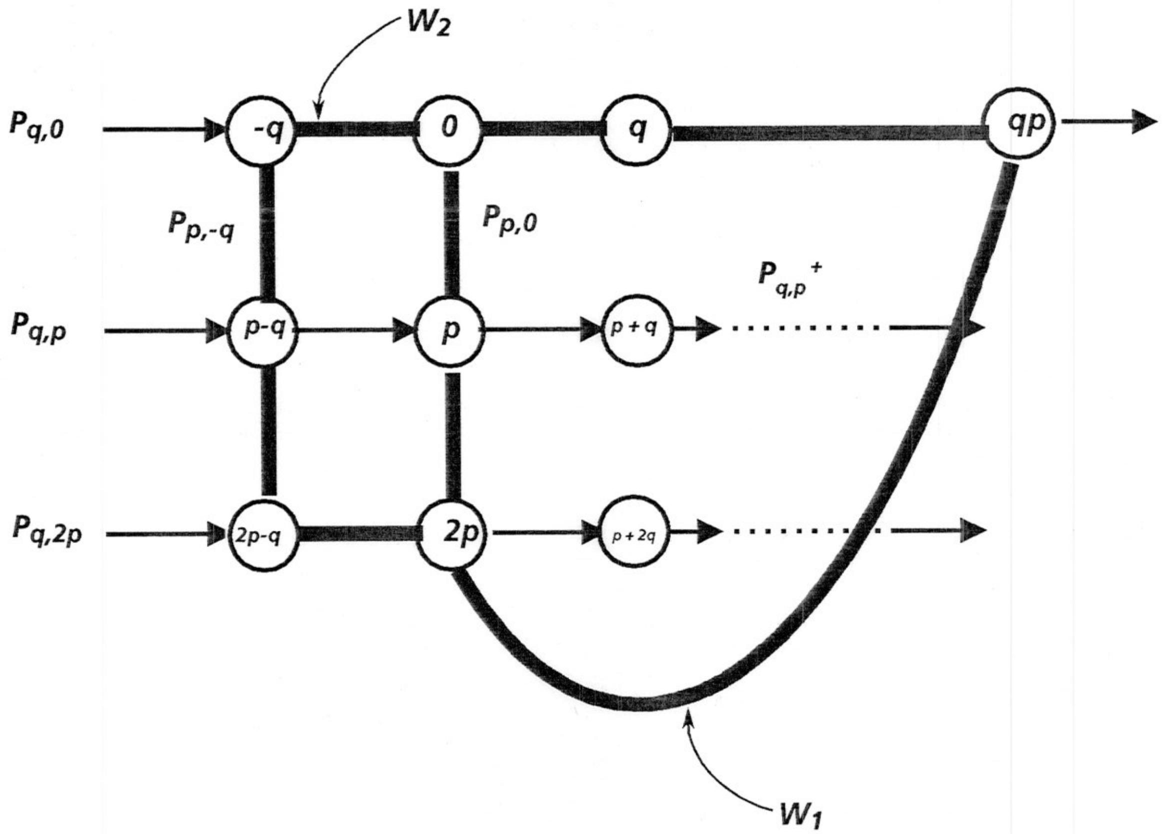


Figure 5. In the case of $1<p<q$, there is no VAP-free planar representation. $P_{q,p}^{+}$ should exist in W_1 or W_2, but this implies a VAP.

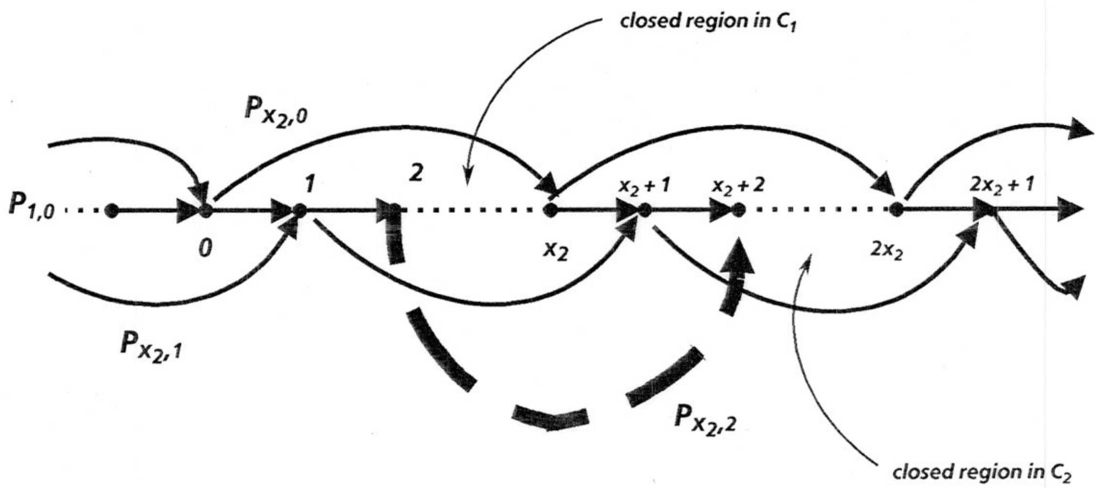


Figure 6. There is no way to connect the node *2* and the node x_2+2 without crossing $P_{x_2,0}$ or $P_{x_2,1}$ as indicated by the wide dotted lines above.

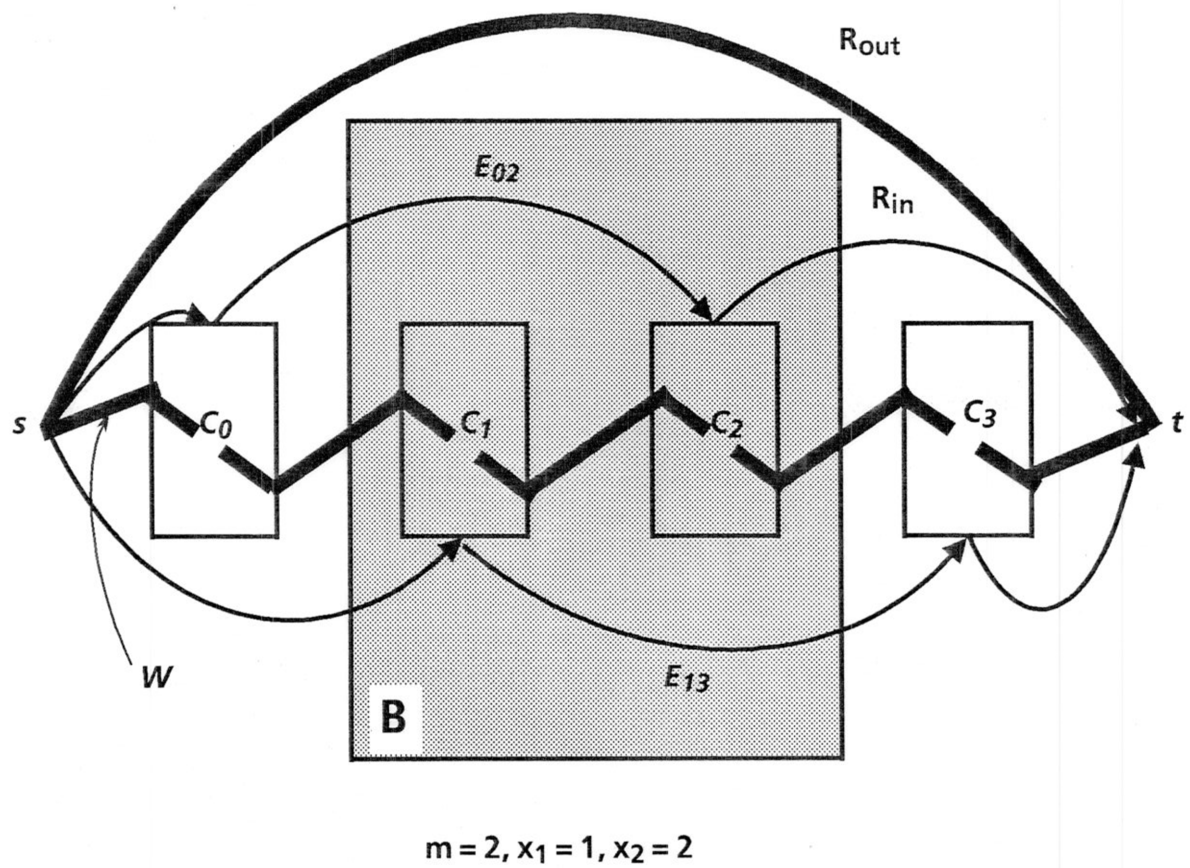


Figure 7. The graph G_f is planar if and only if G^1 has a VAP-free planar representation.

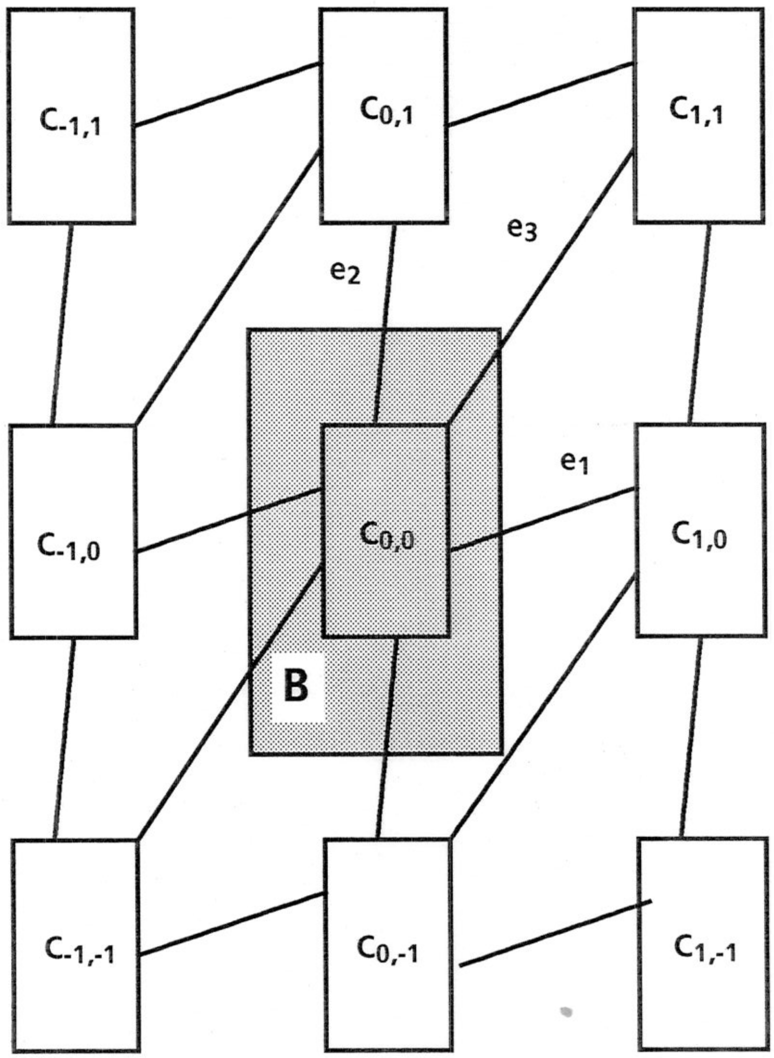


Figure 8. This finite graph G_f is planar if and only if the infinite graph G^2 has a VAP-free planar representation.