A SEMIRING ON CONVEX POLYGONS AND ZERO-SUM CYCLE PROBLEMS

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Abstract

We show that two natural operations on the set of convex polygons form a closed semiring; the two operations are vector summation and convex hull of the union. We then investigate various properties of these operations: for example, the operation of vector summation takes $O(m \log m)$ time where m is the number of edges involved in the operation, while the decomposition of a given convex polygon into two convex polygons (in a sense, the inverse of vector summation) is NP-complete.

Kleene's algorithm applied to this closed semiring solves the problem of determining whether a directed graph with two-dimensional labels has a zero-sum cycle or not. We show that this algorithm runs in polynomial time in the special cases of graphs with one-dimensional labels, BTTSP (Backedged Two-Terminal Series-Parallel) graphs, and graphs with bounded labels. We also investigate the undirected zero-sum cycle problem and the zero-sum *simple* cycle problem.

1. Introduction

In this paper, we show that two natural operations on the set of convex polygons form a closed semiring; the two operations are vector summation and convex hull of the union. We then investigate the time complexity of each operation and its effect on the number of edges of the polygons. The inverse operation of vector summation, known as *decomposition*, has been well studied in [21, 22, 23]. The *decomposability problem* is to determine whether or not a given convex polygon is the summation of two convex polygons. We show that the decomposability problem in two-dimensions is NP-complete, while vector summation takes $O(m \log m)$ time where m is the number of edges in the original convex polygon.

Kleene's algorithm applied to various closed semirings leads to efficient algorithms for a variety of problems: for example, finding the shortest paths for all pairs of nodes [10, 11, 12], converting a finite automaton into a regular expression, and finding the most reliable or largest-capacity path [5]. In this paper we

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use the above closed semiring to solve the *zero-sum cycle problem* in doubly weighted directed graphs.

Doubly weighted graphs, which have a two-dimensional weight on each edge, have been studied by Lawler [6], Dantzig et al. [7] and Reiter [8]. The cost of a path is defined as the sum of weights of edges in the path. The *doubly weighted zero-sum cycle problem* is to find a cycle whose cost in each dimension is 0. In [1 - 4], we saw that certain problems in VLSI applications involving a regular structure can be transformed to problems in two-dimensional infinite graphs consisting of repeated finite graphs. Repeated use of a doubly weighted digraph, called the *static graph* G, forms a *dynamic graph* $G^{\infty,\infty}$. As shown in Fig. 1, each label of the static graph G indicates the differences between the x- and y-coordinates of two connected vertices in $G^{\infty,\infty}$. The absence of a zero-sum cycle in the specified static graph is then necessary and sufficient for the acyclicity of the associated dynamic graph. A two-dimensional regular electrical circuit is associated with a dynamic graph, and acyclicity of the dynamic graph implies that the associated electrical circuit is *valid*; that is, free of an electrical "short circuit" [4].

We focus on the convex hulls of the lengths of all paths between two vertices and apply the two operations above to the set of these convex hulls. We then use the closed semiring defined by these two operations to solve the doubly weighted zero-sum cycle problem. We show that this algorithm runs in polynomial time in the special cases of bounded label graphs, BTTSP graphs (the Backedged Two-Terminal Series-Parallel graphs), and graphs with one-dimensional labels. The *1-bounded* graphs, whose labels are 0, 1, or -1, arise in VLSI applications where the interconnections between regular basic cells are made locally. The BTTSP graphs are an extension of the class of *Two-Terminal Series-Parallel* [15, 16, 17, 18, 19, 20]. Whether the zero-sum cycle problem for general graphs is in P remains open.

Finally, we also discuss variations of the zero-sum cycle problem, the undirected case, and the zero-sum *simple* cycle problem.

2. Two operations and a semiring

In this section, we define two operations on the set of convex polygons which form a closed semiring, defined in [25] as follows:

Definition. A *semiring* is the system (S, $+$, $\cdot$, 0, 1), where S is a set of elements, and $+$ and $\cdot$ are binary operations on S, satisfying the following three properties:

1) (S, $+$, 0) is a commutative monoid; that is, it is closed under $+$, $+$ is commutative and associative, and 0 is an identity.

2) (S, $\cdot$, 1) is a monoid; that is, it is closed under $\cdot$, $\cdot$ is associative, and 1 is an identity.

3) $\cdot$ distributes over + and 0 is an annihilator, i.e., $a \cdot 0 = 0 \cdot a = 0$.

A *semiring* (S, +, $\cdot$, 0, 1) is a *closed semiring*, if in addition infinite sums exist, i.e. with every family $\{ a_i, i \in I \}$ of elements of S with countable (finite or infinite) index set I, there is an associated element $\sum_{i \in I} a_i$, its sum. Infinite sums must satisfy the following properties:

4) For finite non-empty index set $I = \{ i_1, i_2, \ldots, i_k \}$,

$$\sum_{i \in I} a_i = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$$

and for empty index set $I = \varnothing$,

$$\sum_{i \in \varnothing} a_i = 0.$$

5) The result of summation does not depend on the ordering of the factors, i.e. for every index set I and every partition $\{ I_j : j \in J \}$ of I such that

$$\bigcup_{j \in J} I_j = I \quad \text{and} \quad I_j \cap I_k = \varnothing \text{ for } j \neq k,$$

we have

$$\sum_{i \in I} a_i = \sum_{j \in J} (\sum_{i \in I_j} a_i).$$

6) Multiplication distributes over infinite sums, i.e.

$$(\sum_{i \in I} a_i) \cdot (\sum_{j \in J} b_j) = \sum_{i \in I} (\sum_{j \in J} a_i \cdot b_j).$$

Now we define our closed semiring. Let S be the set of all convex polygons. That is, $S = \{ \alpha^+ \mid \alpha \in 2^{Z \times Z} \}$, where α^+ indicates the convex hull of α. (Notice that this definition allows polytopes with an infinite number of edges, extended to infinity, but does not allow curves.) We conventionally denote an element in S by a lower-case Greek letter. We regard a point or a line segment as an element of S.

Definition. For any two sets $\alpha, \beta \in S$, we define the new set called *β-shifted* α as follows:

$$\alpha_\beta = \{ (x, y) \mid \textit{there exist elements } (a_x, a_y) \in \alpha \text{ and } (b_x, b_y) \in \beta \; \textit{such that } x = a_x + b_x,\ y = a_y + b_y \}.$$

□

From the definition, $\alpha_\beta = \beta_\alpha$. Let $\varepsilon = \{ (0, 0) \} \in S$. Let $\varnothing$ be the empty set.

Definition. We define a *system* (S, +, $\cdot$, $\varnothing$, ε) as follows: for $\alpha, \beta \in S$, we define

$$\alpha + \beta = (\alpha \cup \beta)^+ \quad \text{and} \quad \alpha \cdot \beta = (\alpha_\beta)^+.$$

That is, $\alpha + \beta$ is the convex hull of the union of α and β, while $\alpha \cdot \beta$ is the convex hull of β-shifted α. By convention, we define $\alpha \cdot \emptyset = \emptyset \cdot \alpha = \emptyset$. A convex polygon $\alpha \cdot p$ is called a *p-copy* of α or simply a *copy* of α when p is a point. □

Note that as we show later (Corollary 3.2 in Section 3), α_β is itself a convex polygon when α and β are convex polygons. Therefore $\alpha \cdot \beta = (\alpha_\beta)^+ = \alpha_\beta$ for any $\alpha, \beta \in$ S. Fig. 2 shows an example of a β-shifted α, that is, $\alpha \cdot \beta$. Since $\alpha_\beta = \beta_\alpha$, the $\cdot$ operation is commutative.

Definition Let I be a countable (finite or infinite) index set. Let $\alpha_i \in$ S for all $i \in I$. Then we define the sum $\sum_{i \in I} \alpha_i$ as follows:

$$\sum_{i \in I} \alpha_i = (\bigcup_{i \in I} \alpha_i)^+.$$

□

Since $\bigcup_{i \in I} \alpha_i$ exists and is unique, its convex hull $\sum_{i \in I} \alpha_i$ exists and is unique. Note that α_i is the convex hull of some set in $2^{Z \times Z}$, and thus every vertex of $\bigcup_{i \in I} \alpha_i$ is in $2^{Z \times Z}$. Hence $\sum_{i \in I} \alpha_i \in$ S, and the sum above is well defined.

Lemma 2.1. Let $\alpha, \beta \in 2^{Z \times Z}$. Then

$$(\alpha^+ \cup \beta)^+ = (\alpha \cup \beta)^+.$$

Proof. Since $\alpha \subset \alpha^+$, we have

$$\alpha \cup \beta \subset \alpha^+ \cup \beta \subset (\alpha^+ \cup \beta)^+. \tag{2.1}$$

Since the right-hand-side of (2.1) is a convex polygon, it contains the convex hull of the left-hand-side of (2.1). Thus

$$(\alpha \cup \beta)^+ \subset (\alpha^+ \cup \beta)^+. \tag{2.2}$$

Conversely we prove the opposite direction of (2.2). Note that $\alpha^+ \subset (\alpha \cup \beta)^+$ and thus $\alpha^+ \cup \beta \subset (\alpha \cup \beta)^+$. Therefore by the same argument as above,

$$(\alpha^+ \cup \beta)^+ \subset (\alpha \cup \beta)^+. \tag{2.3}$$

□

In the same way as above, we have the following lemma:

Lemma 2.2. Let I be a countable index set. Let $\alpha_i \in 2^{Z \times Z}$ for all $i \in I$. Then we have

$$(\bigcup_{i \in I} \alpha_i^+)^+ = (\bigcup_{i \in I} \alpha_i)^+. \tag{2.4}$$

Proof. Let A be the left-hand-side of (2.4), and B the right-hand-side. Since $\alpha_i \subset A$ for all $i \in I$, we have $\bigcup_{i \in I} \alpha_i \subset A$. Since A is a convex hull, we have

$$B = (\bigcup_{i \in I} \alpha_i)^+ \subset A.$$

Note that $\alpha_i \subset B$ and thus $\alpha_i{}^+ \subset B$ for all $i \in I$. Therefore $\bigcup_{i \in I} (\alpha_i^+) \subset B$, and moreover since B is a convex polygon,

$$A = (\bigcup_{i \in I} \alpha_i^+)^+ \subset B.$$

□

Theorem 2.1. The system (S, +, · , ∅, ε) is a closed semiring.

Proof. We show that the system (S, +, · , ∅, ε) satisfies the six properties of a closed semiring.

1) (S, +, ∅) is a commutative monoid. Since $\alpha + \beta$ is a convex polygon, the set S is closed under +. Let $\alpha, \beta, \gamma \in$ S. Then

$$\begin{aligned}
(\alpha + \beta) + \gamma &= ((\alpha \cup \beta)^+ \cup \gamma)^+ \\
&= ((\alpha \cup \beta) \cup \gamma)^+ && \text{(from Lemma 2.1)} \\
&= (\alpha \cup (\beta \cup \gamma))^+ \\
&= (\alpha \cup (\beta \cup \gamma)^+)^+ && \text{(from Lemma 2.1)} \\
&= (\alpha \cup (\beta + \gamma))^+ && \text{(from the definition)} \\
&= \alpha + (\beta + \gamma). && \text{(from the definition)}
\end{aligned}$$

Hence + is associative. Since

$$\alpha + \beta = (\alpha \cup \beta)^+ = (\beta \cup \alpha)^+ = \beta + \alpha,$$

+ is commutative. Since

$$\alpha + \varnothing = (\alpha \cup \varnothing)^+ = \alpha^+ = \alpha$$

for any $\alpha \in$ S, ∅ is the identity for +.

2) (S, ·, ε) is a monoid. Since $\alpha \cdot \beta$ is a convex polygon, the set S is closed under ·.

$$\begin{aligned}
\alpha \cdot (\beta \cdot \gamma) &= \alpha_{(\beta_\gamma)} && \text{(from the defintion and Corollary 3.2)} \\
&= \{ (x, y) \mid \textit{there exist } (a_x, a_y) \in \alpha, \\
&\qquad (b_x, b_y) \in \beta, \text{ and } (g_x, g_y) \in \gamma \textit{ such that} \\
&\qquad x = a_x + b_x + g_x,\ y = a_y + b_y + g_y \} \\
&= (\alpha_\beta)_\gamma && \text{(using symmetry)}
\end{aligned}$$

$$= (\alpha \cdot \beta) \cdot \gamma.$$

Thus $\cdot$ is associative. Since

$$\alpha \cdot \varepsilon = \{ (x, y) \mid \textit{there exists } (a_x, a_y) \in \alpha \textit{ such that } x = a_x + 0 = a_x, y = a_y + 0 = a_y. \}$$
$$= \varepsilon \cdot \alpha = \alpha$$

for any $\alpha \in$ S, ε is the identity for $\cdot$.

3) $\cdot$ distributes over $+$. Let α, β, $\gamma \in$ S be convex polygons. Since

$$\alpha \cdot \beta \subset \alpha \cdot (\beta + \gamma) \text{ and } \alpha \cdot \gamma \subset \alpha \cdot (\beta + \gamma),$$

we have

$$(\alpha \cdot \beta) \cup (\alpha \cdot \gamma) \subset \alpha \cdot (\beta + \gamma). \tag{2.5}$$

Since the right-hand-side of (2.5) is a convex polygon, we have

$$[(\alpha \cdot \beta) \cup (\alpha \cdot \gamma)]^+ \subset \alpha \cdot (\beta + \gamma). \tag{2.6}$$

Note that the left-hand-side of (2.6) is $(\alpha \cdot \beta) + (\alpha \cdot \gamma)$ from the definition. Thus we proved

$$(\alpha \cdot \beta) + (\alpha \cdot \gamma) \subset \alpha \cdot (\beta + \gamma). \tag{2.7}$$

Conversely we prove the opposite direction of (2.7). Let x be a point in $\alpha \cdot (\beta + \gamma)$. Then x can be represented by $a \cdot b$ (treating a and b as single point polygons) such that $a \in \alpha$ and b is on the line segment $\overline{b_1, b_2}$ where $b_1, b_2 \in \beta \cup \gamma$. If b_1 and b_2 are both in β (or γ), $x = a \cdot b$ is in $\alpha \cdot \beta$ (or $\alpha \cdot \gamma$). Otherwise, without loss of generality, we can assume that $b_1 \in \beta$ and $b_2 \in \gamma$. Then the point $x = a \cdot b$ is on the line segment $\overline{a \cdot b_1, a \cdot b_2}$ where $a \cdot b_1 \in \alpha \cdot \beta$ and $a \cdot b_2 \in \alpha \cdot \gamma$. Thus

$$x \in (\alpha \cdot \beta) + (\alpha \cdot \gamma).$$

Therefore

$$\alpha \cdot (\beta + \gamma) \subset (\alpha \cdot \beta) + (\alpha \cdot \gamma). \tag{2.8}$$

From (2.7) and (2.8), we have

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma).$$

Thus $\cdot$ distributes over $+$. Note that we can also prove that $\cdot$ distributes over finite sums by induction.

Since

$$\alpha + \alpha = (\alpha \cup \alpha)^+ = \alpha^+ = \alpha,$$

$+$ is idempotent.

4) Let $I = \{ i_1, i_2, \ldots, i_k \}$ be a finite non-empty index set. Let $\alpha_i \in S$ for all $i \in I$. Then we prove

$$\sum_{i \in I} \alpha_i = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} \tag{2.9}$$

by induction on k. When $k = 1$, (2.9) holds. Assume that (2.9) holds for the numbers less than k.

$$\begin{aligned} & \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{k-1}} + \alpha_{i_k} \\ &= (\sum_{i \in I - \{ i_k \}} \alpha_i) + \alpha_{i_k} \qquad \text{(from the induction hypothesis)} \\ &= (\bigcup_{i \in I - \{ i_k \}} \alpha_i)^+ + \alpha_{i_k} \qquad \text{(from the definition of } \textstyle\sum \text{)} \\ &= ((\bigcup_{i \in I - \{ i_k \}} \alpha_i)^+ \cup \alpha_{i_k})^+ \\ &= ((\bigcup_{i \in I - \{ i_k \}} \alpha_i) \cup \alpha_{i_k})^+ \qquad \text{(from Lemma 2.1)} \\ &= (\bigcup_{i \in I} \alpha_i)^+ = \sum_{i \in I} \alpha_i \end{aligned}$$

For empty index set $I = \varnothing$, we have

$$\sum_{i \in \varnothing} \alpha_i = \varnothing.$$

5) Let I be an arbitrary index set I and $\{ I_j : j \in J \}$ be an arbitrary partition of I such that

$$\bigcup_{j \in J} I_j = I \quad \text{and} \quad I_j \cap I_k = \varnothing \ \text{ for } \ j \neq k.$$

Then we have

$$\begin{aligned} \sum_{j \in J} (\sum_{i \in I_j} \alpha_i) &= \sum_{j \in J} (\bigcup_{i \in I_j} \alpha_i)^+ \\ &= (\bigcup_{j \in J} (\bigcup_{i \in I_j} \alpha_i)^+)^+ \qquad \text{(from the definition of } \textstyle\sum \text{)} \\ &= (\bigcup_{j \in J} (\bigcup_{i \in I_j} \alpha_i))^+ \qquad \text{(from Lemma 2.2)} \\ &= (\bigcup_{i \in I} \alpha_i)^+ = \sum_{i \in I} \alpha_i . \end{aligned}$$

Thus the result of summation does not depend on the ordering of the factors.

6) Let $\beta \in S$. Let $\alpha_i \in S$ for $i = 1, 2, \cdots$. Let

$$Z_\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \cdots .$$

Let

$$Z_{\alpha \cdot \beta} = \alpha_1 \cdot \beta + \alpha_2 \cdot \beta + \cdots + \alpha_i \cdot \beta + \cdots .$$

Then we prove that

$$\beta \cdot Z_{\alpha} = Z_{\alpha \cdot \beta}. \tag{2.10}$$

We first prove that $\beta \cdot Z_{\alpha} \subset Z_{\alpha \cdot \beta}$. Let $p = b \cdot x$ be an arbitrary point in $\beta \cdot Z_{\alpha}$ with $b \in \beta$ and $x \in Z_{\alpha}$. If there exists a finite set of indexes $J = \{ j_1, j_2, \ldots, j_m \}$ such that

$$x \in \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m},$$

then

$$\begin{aligned} p = b \cdot x &\in \beta \cdot (\alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m}) \\ &= \beta \cdot \alpha_{j_1} + \beta \cdot \alpha_{j_2} + \cdots + \beta \cdot \alpha_{j_m} \quad \text{(from 3))} \\ &\subset \beta \cdot \alpha_1 + \beta \cdot \alpha_2 + \cdots + \beta \cdot \alpha_i + \cdots = Z_{\alpha \cdot \beta}. \end{aligned}$$

If x is not in the sum of a finite number of α_i's, then x must be the limit point of a sequence of points, each of which is in some α_i; that is, there exists an infinite set of indexes $J = \{ j_1, j_2, \ldots, j_i, \ldots, \}$ such that

$$x = \lim_{i \to \infty} x_{j_i} \quad \text{where} \quad x_{j_i} \in \alpha_{j_i}.$$

Then

$$\begin{aligned} p = b \cdot x &= b \cdot \lim_{i \to \infty} x_{j_i} \\ &= \lim_{i \to \infty} b \cdot x_{j_i} \in \lim_{i \to \infty} \beta \cdot \alpha_{j_i} \\ &\subset \beta \cdot \alpha_{j_1} + \beta \cdot \alpha_{j_2} + \cdots + \beta \cdot \alpha_{j_i} + \cdots \\ &\subset \beta \cdot \alpha_1 + \beta \cdot \alpha_2 + \cdots + \beta \cdot \alpha_i + \cdots = Z_{\alpha \cdot \beta} \end{aligned}$$

Therefore $\beta \cdot Z_{\alpha} \subset Z_{\alpha \cdot \beta}$.

We now prove the converse; that is, $Z_{\alpha \cdot \beta} \subset \beta \cdot Z_{\alpha}$. Let p be an arbitrary point in $Z_{\alpha \cdot \beta}$. If there exists a finite set of indexes $J = \{ j_1, j_2, \ldots, j_m \}$ such that

$$p \in \beta \cdot \alpha_{j_1} + \beta \cdot \alpha_{j_2} + \cdots + \beta \cdot \alpha_{j_m}.$$

Then

$$p \in \beta \cdot (\alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m}) \subset \beta \cdot Z_{\alpha}. \quad \text{(from 3))}$$

If not, there exists an infinite set of indexes $J = \{ j_1, j_2, \ldots, j_i, \ldots, \}$ and $p_i = b_i \cdot x_i$ with $b_i \in \beta$ and $x_i \in \alpha_{j_i}$ such that

$$p = \lim_{i \to \infty} p_i = \lim_{i \to \infty} b_i \cdot x_i.$$

Let $\gamma_i = (\bigcup_{k=1}^{i} x_k)^+$. Note that $\gamma_i \subset \gamma_{i+1}$ for all $i \in Z$. We now have

$$p = \lim_{i \to \infty} b_i \cdot x_i \in \lim_{i \to \infty} \beta \cdot x_i \qquad (\text{ since } b_i \in \beta\ )$$

$$\subset \beta \cdot x_1 + \beta \cdot x_2 + \cdots + \beta \cdot x_i + \cdots$$

$$\subset \beta \cdot \gamma_1 + \beta \cdot \gamma_2 + \cdots + \beta \cdot \gamma_i + \cdots \qquad (\text{ since } x_i \in \gamma_i\ )$$

$$= \lim_{i \to \infty} \beta \cdot \gamma_i \qquad (\text{ since } \gamma_i \subset \gamma_{i+1}\ )$$

$$= \beta \cdot \lim_{i \to \infty} \gamma_i = \beta \cdot \lim_{i \to \infty} (\bigcup_{k=1}^{i} x_i)^+$$

$$\subset \beta \cdot (x_1 + x_2 + \cdots + x_i + \cdots) \subset \beta \cdot Z_\alpha .$$

Therefore we have $Z_{\alpha \cdot \beta} = \beta \cdot Z_\alpha$.

Now let I, J be countable index sets and $\alpha_i \in I$, $\beta_j \in J$ for $i \in I$ and $j \in J$. Thus

$$(\sum_{i \in I} \alpha_i) \cdot (\sum_{j \in J} \beta_j) = \sum_{i \in I} (\alpha_i \cdot (\sum_{j \in J} \beta_j)) \quad (\text{ using (2.10) })$$

$$= \sum_{i \in I} (\sum_{j \in J} \alpha_i \cdot \beta_j). \quad (\text{ using (2.10) })$$

Thus multiplication distributes over infinite sums. □

Having established that the structure (S, +, · , ∅, ε) is a closed semiring, we can apply Kleene's algorithm to solve certain problems related to paths in a graph ([12, 25]). With this goal in mind we next investigate the basic properties of the operations + and ·.

3. Some properties of the · operation

In this section, we investigate the properties of the · operation.

Definition. Let α_E be the set of edges of the convex polygon α, and α_V the set of its vertices. □

Definition. Let α be a convex polygon. Let l be an edge of α or a line which does not intersect α. Then we regard l as the oriented line with respect to α and define its direction, denoted by $\theta_l(\alpha)$, in the range $-3/2\,\pi < \theta_e \le 1/2\,\pi$ such that α lies on the right-hand-side of l when we traverse l in its positive direction. Unless specified, θ_e means $\theta_e(\alpha)$ for an edge $e \in \alpha_E$. We regard $e \in \alpha_E$ as a *vector* **e** with the direction of $\theta_e(\alpha)$.

We say that two edges $e \in \alpha_E$ and $f \in \beta_E$ are *equivalent* when $\theta_e(\alpha) = \theta_f(\beta)$. Equivalent edges are said to be *twins* when their lengths are equal. If there are no equivalent edges in $\alpha_E \cup \beta_E$, we say that α and β are *independent.* □

Definition. Let $\alpha_{vector} = \{ \mathbf{e} \mid e \in \alpha_E \}$. By convention, we define the following special cases: When α is either a point or the entire plane, we regard α as a special symbol and define $\alpha_{vector} = \{ \alpha \}$. When α is a line segment e, we define $\alpha_{vector} = \{ +\theta_e, -\theta_e \}$ where $\theta_e = \theta_e(\alpha)$. □

Definition. Let $A = \{ \alpha_i \}$ be a set of convex polygons. We define

$$| A | = | \bigcup_{\alpha_i \in A} (\alpha_i)_{vector} |$$

$$= \textit{the number of distinct vectors in } \bigcup_{\alpha_i \in A} (\alpha_i)_{vector}$$

We also write $| A | = | \alpha |$ when A has the single element α. □

Now we have the following lemma about the relationship between two consecutive edges of a convex polygon and their directions.

Lemma 3.1. Let e and f be two consecutive edges of a convex polygon α in clockwise order. Then

$$\begin{cases} \theta_f < \theta_e \text{ or } (\theta_e + \pi) < \theta_f & \text{if } \theta_e < -1/2\,\pi \\ (\theta_e - \pi) < \theta_f < \theta_e & \text{if } \theta_e \geq -1/2\,\pi. \end{cases}$$

Proof. From the definition, f lies in the right-half plane of e. Hence we have the desired result. □

Corollary 3.1. Let $\alpha_E = \{ e_1, e_2, \ldots, e_m \}$ be the edges of a convex polygon α in clockwise order. Let θ_{e_1} be the maximum of $\{ \theta_{e_i} \}$. Then

$$\theta_{e_1} > \theta_{e_2} > \cdots > \theta_{e_m}.$$

We call this order the *canonical* order. The set α_E is called the *canonical edge set* when the elements of α_E are ordered canonically.

Proof. This is clear from Lemma 3.1. □

Now we analyze how the $\cdot$ operation affects the number of distinct vectors.

Theorem 3.1. Suppose two convex polygons $\alpha, \beta \in S$ are not points. Then for every $e \in \alpha_E \cup \beta_E$, there exists the equivalent edge f in $(\alpha \cdot \beta)_E$; that is, $\theta_f = \theta_e$. This enables us to define a function $f = \varphi (e)$ from $\alpha_E \cup \beta_E$ to $(\alpha \cdot \beta)_E$. Moreover the function φ is onto.

In Fig. 2, the edges with the same number are equivalent.

Before proving Theorem 3.1, we need some definitions. Given $\alpha, \beta \in S$ and $w = (w_x, w_y) \in \beta_V$, we define the set α_w to be the set of w-copies of vertices in α; that is,

$$\alpha_w = \{ (x, y) \mid \textit{there exists an element } (a_x, a_y) \in \alpha_V \textit{ such that}$$
$$x = a_x + w_x, y = a_y + w_y \}.$$

Then we have $\alpha_\beta = (\bigcup_{w \in \beta_V} \alpha_w)^+$. Let $s \in (\alpha \cdot \beta)_V$. Then there exist $v \in \alpha_V$

and $w \in \beta_V$ such that $s = v \cdot w \in \beta_v \cap \alpha_w$. Hence we denote s by $p_{v,w}$.

Proof of Theorem 3.1. Let $\alpha, \beta \in S$ with $\alpha_V = \{ v_1, v_2, \ldots, v_m \}$ and $\beta_V = \{ w_1, w_2, \ldots, w_n \}$. Let $\gamma = \alpha \cdot \beta$. We assume that v_i and w_j are labeled in clockwise order and the indexes of vertices v_i (w_j) are interpreted mod m (mod n). From the above discussion, we denote a vertex in γ_V by p_{ij} with $p_{ij} = v_i \cdot w_j$ where $v_i \in \alpha_V$ and $w_j \in \beta_V$.

Suppose that $e \in \alpha_E \cup \beta_E$. Then we prove that there exists an edge $f \in (\alpha \cdot \beta)_E$ such that $\theta_f = \theta_e$. Note that there is no pair of equivalent edges in γ_E, so there is the unique edge $f \in \gamma_E$ such that $\theta_f = \theta_e$. Assume that $e = (v_i, v_{i+1}) \in \alpha_E$. From Corollary 3.1, as shown in Fig. 3a, there exists some k such that θ_e is between θ_{e_k} and $\theta_{e_{k+1}}$ where e_k and e_{k+1} are edges in β_E in clockwise order. Let w_j be the common vertex of e_k and e_{k+1}, and let e' be the vector from w_j such that $\theta_{e'} = \theta_e$. Note that as shown in Fig. 3a, any point $v_x \in \alpha_V$ is in the right-hand-side of the edge e, and any point $w_y \in \beta_V$ is also in the right-hand-side of the edge e'. Thus as shown in Fig. 3b, any point $p_{xy} = v_x \cdot w_y$ in γ is also in the right-hand-side of the edge $f = w_j \cdot e$. Therefore f is the edge of γ such that $\theta_f = \theta_e$.

Conversely suppose that $f = e_{ij,kl} = (p_{ij}, p_{kl})$ is an edge of γ. Then we prove that there is an edge $e \in \alpha_E \cup \beta_E$ such that $\theta_e = \theta_f$. Let

$$d_x = k - i \pmod{m} \text{ and } d_y = l - j \pmod{n}.$$

Then we claim that

$$(d_x = 0, d_y = \pm 1) \text{ or } (d_x = \pm 1, d_y = 0).$$

Assume that the claim above does not hold. If $d_x = 0$ and $| d_y | \geq 2$, then the edge $e_{ij,kl} = e_{ij,il} = (p_{ij}, p_{il})$ is a diagonal line of the convex hull β_{v_i} as shown in Fig. 3c. Hence the line segment $e_{ij,kl}$ cannot be an edge of γ. In the same way, the line segment $e_{ij,kl}$ such that $| d_x | \geq 2$ and $d_y = 0$ cannot be an edge of γ. Now assume that $d_x \geq 1$ and $d_y \geq 1$. In this case the line segment $e_{ij,kl}$ is a diagonal line of the parallelogram P consisting of four vertices p_{ij}, p_{il}, p_{kl}, and p_{kj} (P is shown by double lines in Fig. 3c.) However each edge of P is a diagonal line of a convex hull α_{w_j}, α_{w_l}, β_{v_i} or β_{v_k}, thus the line segment $e_{ij,kl}$ cannot be an edge of γ. Thus $(\alpha \cdot \beta)_E$ consists of

$$(p_{ij}, p_{i \pm 1,j}) = (v_i, v_{i \pm 1}) \cdot w_j \quad \text{or} \quad (p_{ij}, p_{i,j \pm 1}) = v_i \cdot (w_j, w_{j \pm 1}).$$

Therefore we have

$$e = (v_i, v_{i \pm 1}) \in \alpha \quad \text{or} \quad e = (w_j, w_{j \pm 1}) \in \beta$$

and e satisfies $\theta_e = \theta_f$. □

Corollary 3.2. Let α, β be convex polygons. Then $\alpha \cdot \beta = \alpha_\beta = \beta_\alpha$.

Proof. From the proof of Theorem 3.1, we know that every edge in $\alpha \cdot \beta$ is either $e \cdot b$ or $a \cdot f$ where $e \in \alpha_E$, $b \in \beta_V$, $f \in \beta_E$, and $a \in \alpha_V$. Therefore

$$\alpha \cdot \beta \subset \alpha_\beta \subset (\alpha_\beta)^+ = \alpha \cdot \beta.$$

Hence $\alpha \cdot \beta = \alpha_\beta = \beta_\alpha$. □

Corollary 3.3. Suppose two convex polygons $\alpha, \beta \in$ S are not points. Then the order of the canonical edge set $(\alpha \cdot \beta)_E$ is obtained by sorting the set $\{\theta_e \mid e \in \alpha_E \cup \beta_E\}$. Thus $\alpha \cdot \beta$ can be computed in $O(n \log n)$ steps where $n = |\alpha \cdot \beta|$.

Proof. From Theorem 3.1, every edge e in $\alpha \cdot \beta$ has an associated edge f in $\alpha_E \cup \beta_E$ such that $\theta_f = \theta_e$. Thus the order for the canonical edge set $(\alpha \cdot \beta)_E$ can be obtained by sorting $\{\theta_f \mid f \in \alpha_E \cup \beta_E\}$ in decreasing order. □

Corollary 3.4. Let $\alpha_1, \alpha_2, \ldots,$ and α_n be convex polygons which are not points. Then we have an onto function φ

$$\text{from } (\alpha_1)_E \cup (\alpha_2)_E \cup \cdots \cup (\alpha_n)_E \text{ to } (\alpha_1 \cdot \alpha_2 \cdots \alpha_n)_E$$

such that

$$\theta_{\varphi(e)} = \theta_e \text{ for any } e \in (\alpha_1)_E \cup (\alpha_2)_E \cup \cdots \cup (\alpha_n)_E.$$

Proof. By induction on n and Theorem 3.1. □

Theorem 3.2. For any $\alpha, \beta \in$ S, we have

$$|\alpha \cdot \beta| \le |\{\alpha, \beta\}| \le |\alpha| + |\beta|.$$

Proof. First we prove that $|\alpha \cdot \beta| \le |\{\alpha, \beta\}|$. If neither α nor β is a point, this is clear from Theorem 3.1. The equality holds if and only if α and β are independent. If either α or β is a point, without loss of generality, we can assume that α is a point p, and then

$$|\alpha \cdot \beta| = |p \cdot \beta| = |\beta| \le |\{\alpha, \beta\}|.$$

Since $|\{\alpha, \beta\}|$ is the number of distinct vectors in $\alpha \cup \beta$, we have

$$|\{\alpha, \beta\}| \le |\alpha| + |\beta|.$$

□

Definition. A convex polygon α is said to be *decomposable* if and only if there exist two convex polygons β and γ such that $\alpha = \beta \cdot \gamma$ and neither β nor γ is a point and there are no equivalent edges in β_E and γ_E. If a convex polygon α is not decomposable, we say α is *irreducible*. □

The decomposability problem has been studied in [21, 22, 23]. Note that the definition of the decomposability is slightly different here. In [21, 22, 23] the authors allow equivalent edges, while we don't. For example, the decomposition shown in Fig. 4a is not allowed by us, but is allowed in [21, 22, 23].

Note that a decomposition into irreducible convex polygons is not necessarily unique. Fig. 4b shows a convex polygon α decomposed into irreducible convex polygons in two ways: $\alpha = \beta_1 \cdot \beta_2 \cdot \beta_3$ and $\alpha = \gamma_1 \cdot \gamma_2$.

The following theorem shows that decomposition of a convex polygon is in general difficult.

Theorem 3.3. The problem determining whether a given convex polygon α is decomposable or not is NP-complete.

Before proving the theorem, we state the problem in another way by regarding the edges of the convex polygon as vectors. Let $\alpha \in$ S be a convex polygon. We express a vector $\mathbf{e} \in \alpha_{vector}$ by two displacements (e_x, e_y); that is, a vector $\mathbf{e}$ is the same vector as (0, 0) $\cdot$ (e_x, e_y). We then formulate our problem DC (the decomposability problem) as follows:

Instance $\mathbf{I_{DC}}$: Let I be a finite index. A set of vectors $\{ \mathbf{e_i} \mid i \in I \}$ such that $\mathbf{e_i} \neq \mathbf{e_j}$ for any i, $j \in I$ with $i \neq j$ and $\sum_{i \in I} \mathbf{e_i} = (0, 0)$.

Question: Is there a proper subset J of I such that $\sum_{j \in J} \mathbf{e_j} = (0, 0)$?

To see that this formulation is equivalent to the decomposability problem, assume first that $\sum_{i \in I} \mathbf{e_i} = (0, 0)$ and $\sum_{j \in J} \mathbf{e_j} = (0, 0)$ where J is a proper subset of I. Note that $\sum_{i \in I} \mathbf{e_i} = (0, 0)$ means that the edges $\{ \mathbf{e_i} \mid i \in I \}$ form a closed path. From Corollary 3.1, the edges $\{ \mathbf{e_i} \mid i \in I \}$ in canonical order form a convex polygon (say α_I). In the same way, the edges $\{ \mathbf{e_j} \mid j \in J \}$ in canonical order form a convex polygon α_J, while the edges $\{ \mathbf{e_j} \mid j \in \bar{J} \}$ in canonical order form a convex polygon $\alpha_{\bar{J}}$. From Corollary 3.1, there exists a point p such that $p \cdot \alpha_J \cdot \alpha_{\bar{J}} = \alpha_I$.

Conversely assume that a convex polygon α is decomposed into two convex polygons β and γ; that is, $\alpha = \beta \cdot \gamma$. Let $\alpha_E = \{ \mathbf{e_i} \mid i \in I \}$. Then α_E is an instance of I_{DC}, since $\mathbf{e_i}$'s are different from each other and $\sum_{i \in I} \mathbf{e_i} = (0, 0)$. From the definition of decomposition, there are no equivalent edges in β_E and γ_E. Thus from Corollary 3.1, every edge in $\beta_E \cup \gamma_E$ can be associated with its twin edge in α_E. Let $J \subset I$ be a subset of indexes such that $\mathbf{e_j} \in \alpha_E$, $j \in J$ has a twin edge in β_E. From the definition of decomposition, J is a proper subset of I. Then $\sum_{i \in J} \mathbf{e_j} = (0, 0)$. Therefore we can find the solution of the instance I_{DC}.

Hence the above formulation is a valid representation of our problem DC.

Proof of Theorem 3.3. It is obvious that the problem DC is in NP, since we can guess a proper subset $J \subset I$ then we can check whether or not $\sum_{j \in J} \mathbf{e_j} = (0, 0)$ in polynomial time.

We reduce the following variation of the subset sum problem SS_1 to the problem DC.

Problem SS_1: a variation of the subset sum problem.

Instance I_{SS_1}: $\{ a_k \in Z^+ \mid k \in K \}$ where K is a finite index and a_k's are different from each other. $B \in Z^+$.

Question: Is there a subset L of K such that $\sum_{l \in L} a_l = B$?

In order to prove that SS_1 is NP-complete, we can use the same reduction as the reduction from the 3-exact cover problem to the subset sum problem, as shown in [14].

We reduce SS_1 to DC as follows: given the above instance I_{SS_1}, we construct an instance I_{DC} with the property that I_{SS_1} has a solution if and only if I_{DC} has a solution. Let $K = \{ 1, 2, \ldots, n \}$. Let $A = \sum_{k \in K} a_k$. Then $A > 0$. Define a set of vectors

$$\{ \mathbf{e_i} \mid i = 1, 2, \ldots, 2n + 3 \}$$

as follows: Let

$$I = \{ 0, 1, \ldots, 2n + 3 \} \text{ and } M = (n + 1) (A + B).$$

Let

$$\mathbf{e_0} = (-B, 1),$$

$$\mathbf{e_i} = (a_i, 1) \quad \text{for } i = 1, 2, \ldots, n,$$

$$\mathbf{e_{n+i}} = (M, -i) \quad \text{for } i = 1, 2, \ldots, n, n + 1,$$

$$\mathbf{e_{2n+2}} = (-M, 0), \text{ and}$$

$$\mathbf{e_{2n+3}} = -(\sum_{i=0}^{2n+2} \mathbf{e_i}) = (-nM + B - A, n(n + 1)/2).$$

The $\mathbf{e_i}$'s are not equivalent to each other. To see that, note that the only possible equivalent vectors in $\{ \mathbf{e_i} \mid i \in I \}$ are $\mathbf{e_0}$ and $\mathbf{e_{2n+3}}$. Assume that $\mathbf{e_0}$ and $\mathbf{e_{2n+3}}$ are equivalent, that is, there is a positive integer c such that

$$\mathbf{e_{2n+3}} = c\, \mathbf{e_0}.$$

Then

$$(-nM + B - A,\ n(n+1)/2) = (-cB,\ c).$$

Thus we have

$$-nM + B - A = [-n(n+1)B]/2$$

or

$$[n(n+1) - 2]B + [2n(n+1) + 2]A = 0.$$

Note that $n(n+1) - 2 \geq 0$, $B > 0$, $2n(n+1) + 2 > 0$, and $A > 0$. Thus the left-hand-side of the above equation is positive. Therefore we got a contradiction. Hence $\mathbf{e_0}$ and $\mathbf{e_{2n+3}}$ are not equivalent. Therefore the vectors in $\{ \mathbf{e_i} \mid i \in I \}$ are different from each other. Moreover $\sum_{i=0}^{2n+3} \mathbf{e_i} = 0$, and thus $\{ \mathbf{e_i} \mid i \in I \}$ is an instance of I_{DC}.

Suppose first that I_{DC} has a solution; that is, there exists a proper subset $J \subset \{1, 2, \ldots, 2n+3\}$ and $\sum_{j \in J} \mathbf{e_j} = (0, 0)$. We consider the following two subcases: a) $2n + 3 \in \bar{J}$ and b) $2n + 3 \in J$.

a) $2n + 3 \in \bar{J}$: Note that $2n + 2 \in J$, since some $i \in \{1, 2, \ldots, n\}$ and some $n + k \in \{n+1, n+2, \ldots, 2n+1\}$ are in J. Since

$$\mathbf{e_{2n+2}} + \mathbf{e_{n+k}} = (0, -k),$$

there exists subindex $K' \subset \{0, 1, \ldots, n\}$ such that $|K'| = k$ and

$$\Big(\sum_{k \in K'} \mathbf{e_k} \Big) + \mathbf{e_{2n+2}} + \mathbf{e_{n+k}} = (0, 0).$$

Note that only $\mathbf{e_0}$ has negative first coordinate among $\{ \mathbf{e_k} \mid k = 0, 1, \ldots, n \}$. Therefore $0 \in K'$. Thus

$$-B + \sum_{k \in K' - \{0\}} a_k = 0.$$

Therefore I_{SS_1} has a solution.

b) $2n + 3 \in J$: We now take two cases: b.1) $2n + 2 \in J$ and b.2) $2n + 2 \in \bar{J}$.

If b.1) holds then we obtain a contradiction as follows. Since

$$\mathbf{e_{2n+2}} + \mathbf{e_{2n+3}} = \Big(-(n+1)M + B - \sum_{k \in K} a_k,\ n(n+1)/2 \Big),$$

from the definition of M,

$$\{ n+1, n+2, \ldots, 2n+1 \} \subset J.$$

Note that

$$\sum_{i=n+1}^{2n+3} \mathbf{e_i} = \Big(B - \sum_{k \in K} a_k,\ -(n+1) \Big).$$

Since every $\mathbf{e_k}$ for $k = 0, 1, \ldots, n$ has 1 in its second coordinate, we know $\{ 0, 1, \ldots, n \} \in J$. Thus $J = I$. This is a contradiction, since J should be a proper subset of I.

If b.2) $2n + 2 \in \bar{J}$ holds, and suppose that $2n + 1 \in \bar{J}$, then from the definition of M,

$$\{ n + 1, n + 2, \ldots, 2n \} \subset J.$$

Thus

$$\mathbf{e_{2n+3}} + \sum_{i = n+1}^{2n} \mathbf{e_i} = (B - \sum_{k \in K} a_k, 0).$$

Since every second coordinate of $\{ \mathbf{e_i} \mid i = 0, 1, \ldots, n \}$ is 1, we have $\{ 0, 1, \ldots, n \} \subset \bar{J}$. Therefore

$$J = \{ n + 1, n + 2, \ldots, 2n \} \text{ and } B = \sum_{k \in K} a_k.$$

Thus we found a solution of I_{SS_1}.

If $2n + 1 \in J$, then there exists an index $n + k \in \bar{J}$ such that

$$n + k \in \{ n + 1, n + 2, \ldots, 2n \}.$$

Let

$$A = \mathbf{e_{2n+3}} + \sum_{i \in \{ n+1, \ldots, 2n \},\, i \neq n+k} \mathbf{e_i} = (B - \sum_{k \in K} a_k, -n - 1 + k).$$

If $0 \in J$, then

$$\mathbf{e_0} + A = (- \sum_{k \in K} a_k, -n + k).$$

Thus there exists subindex

$$J' \subset \{ 1, 2, \ldots, n \}$$

such that

$$| J' | = n - k \text{ and } - \sum_{k \in K} a_k + \sum_{j \in J'} a_j = 0.$$

This is a contradiction, since a_i's are positive and $| J' | = n - k < n = | K |$ or $| J' | \neq | K |$.

If $0 \in \bar{J}$, then there exists subindex

$$J' \subset \{ 1, 2, \ldots, n \}$$

such that

$$| J' | = n + 1 - k \text{ and } B - \sum_{k \in K} a_i + \sum_{j \in J'} a_j = 0.$$

Therefore we found a solution of I_{SS_1}.

In either case, we have a solution of I_{SS_1}.

Conversely if I_{SS_1} has a solution J in I such that $\sum_{j \in J} a_j = B$, it is obvious that I_{DC} has a solution.

We have therefore reduced a variation of the subset sum problem to the decomposition problem in NP. Thus the decomposition problem is NP-complete. □

4. Some properties of the + and * operations

Next we analyze the effect of the + operation on the number of distinct vectors.

Lemma 4.1. Let $\alpha \in$ S. Let $p_1, p_2, \ldots, p_n$ be points. Then

$$| \alpha + p_1 + p_2 + \ldots + p_n | \leq | \alpha | + n.$$

Proof. This is proved by induction on m. Suppose $n = 1$. If $p_1 \in \alpha$, then $| p_1 + \alpha | = | \alpha |$. Otherwise, at least one edge of α is inside $p_1 + \alpha$. Thus $| p_1 + \alpha | \leq 1 + | \alpha |$. Therefore Lemma 4.1 holds for $n = 1$. Suppose Lemma 4.1 holds for numbers less than n. Let

$$\beta = \alpha + p_1 + p_2 + \ldots + p_{n-1}.$$

Then from the induction hypothesis,

$$| \beta | \leq | \alpha | + (n - 1) \text{ and } | \beta + p_n | \leq | \beta | + 1.$$

Thus

$$| \alpha + p_1 + p_2 + \ldots + p_n | = | \beta + p_n | \leq | \alpha | + n.$$

Hence Lemma 4.1 holds for every $n \geq 1$. □

Theorem 4.1. Let α, $\beta \in$ S. Then

$$| \alpha + \beta | \leq | \alpha | + | \beta |.$$

Proof. Let $\beta_V = \{ p_1, p_2, \ldots, p_n \}$. Note that

$$\alpha + \beta = \alpha + p_1 + p_2 + \ldots + p_n,$$

and $n \leq | \beta |$. Thus from Lemma 4.1,

$$| \alpha + \beta | \leq | \alpha | + n \leq | \alpha | + | \beta |.$$

□

Definition. For a convex polygon α, we define a convex polygon α^m for nonnegative integer m as follows:

1) $\alpha^0 = \varepsilon$.

2) $\alpha^m = \alpha \cdot \alpha^{m-1}$ for $m > 1$.

Since a system (S, +, ·, ∅, ε) is a closed semiring, we can define the convex polygon α^* by $\alpha^0 + \alpha^1 + \cdots = \sum_{i=0}^{\infty} \alpha^i$. □

See Fig. 5 for an example of the * operation. Note that α^* has at most two edges.

Now we analyze an effect of the * operation on a number of distinct vectors.

Lemma 4.2. For two convex polygons α and γ, we have $| \alpha \cdot \gamma^* | \le | \alpha | + 1$.

Proof. If $\gamma^* = \varepsilon$, we have $| \alpha \cdot \gamma^* | = | \alpha |$. If γ^* is the entire plane, we have $| \alpha \cdot \gamma^* | = 1 \le | \alpha |$. If γ^* is a half-line issuing from the origin, then there is at least one edge of α inside $\alpha \cdot \gamma^*$. Thus $| \alpha \cdot \gamma^* | \le | \alpha | + 1$. If γ^* is a line passing through the origin, then $\alpha \cdot \gamma^*$ is a half plane. Thus $| \alpha \cdot \gamma^* | = 1 \le | \alpha |$. Otherwise γ^* has two edges g_1' and g_2'. Let g_1 (g_2) be the support lines at v (w) of the convex polygon α such that $\theta_{g_1}(\alpha) = \theta_{g_1'}(\gamma^*)$ and $\theta_{g_2}(\alpha) = \theta_{g_2'}(\gamma^*)$. If $v = w$, then $| \alpha \cdot \gamma^* | = | \gamma^* | \le 2$. If $v \ne w$, then as shown in Fig. 5, there must be at least one edge of α which is inside $\alpha \cdot \gamma^*$. Thus $| \alpha \cdot \gamma^* | \le | \alpha | + 1$. □

Lemma 4.3. Let $\gamma_i \in$ S for $i = 1, 2, \ldots, n$, $n \ge 2$. Then

$$\gamma_1^* \cdot \gamma_2^* \cdots \gamma_n^* = (\gamma_1 + \gamma_2 + \cdots + \gamma_n)^*.$$

Proof. We prove this by induction on n. We use k for the index of induction. When $k = 2$, we prove that $\gamma_1^* \cdot \gamma_2^* = (\gamma_1 + \gamma_2)^*$. Since

$$\gamma_1^* \subset (\gamma_1 + \gamma_2)^* \text{ and } \gamma_2^* \subset (\gamma_1 + \gamma_2)^*,$$

we have

$$\gamma_1^* \cdot \gamma_2^* \subset (\gamma_1 + \gamma_2)^*. \tag{4.1}$$

Since $\varepsilon \in \gamma_1^* \cdot \gamma_2^*$, we have

$$\begin{aligned} \gamma_1 + \gamma_2 &\subset (\gamma_1 + \gamma_2) \cdot \gamma_1^* \cdot \gamma_2^* \\ &= (\gamma_1 \cdot \gamma_1^*) \cdot \gamma_2^* + \gamma_1 \cdot (\gamma_2 \cdot \gamma_2^*) \quad \text{(using distribution law)} \\ &\subset \gamma_1^* \cdot \gamma_2^*. \end{aligned}$$

Thus

$$(\gamma_1 + \gamma_2)^* \subset \gamma_1^* \cdot \gamma_2^*. \tag{4.2}$$

Hence from (4.1) and (4.2),

$$\gamma_1^* \cdot \gamma_2^* = (\gamma_1 + \gamma_2)^*.$$

Assume that the lemma holds for $k < n$, then

$$\begin{aligned}\gamma_1^* \cdot \gamma_2^* \cdots \gamma_n^* &= (\gamma_1^* \cdot \gamma_2^* \cdots \gamma_{n-1}^*) \cdot \gamma_n^* \\ &= (\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1})^* \cdot \gamma_n^* \quad (\text{using } k = n - 1) \\ &= (\gamma_1 + \gamma_2 + \cdots + \gamma_n)^*. \quad (\text{using } k = 2)\end{aligned}$$

□

Lemma 4.4. Let α, β, and γ be convex polygons. Then

$$\alpha + \beta \cdot \gamma^* = (\alpha + \beta) \cdot \gamma^*.$$

Proof. See Fig. 5. Since $\alpha \subset \alpha \cdot \gamma^*$, we have

$$\alpha + \beta \cdot \gamma^* \subset \alpha \cdot \gamma^* + \beta \cdot \gamma^* = (\alpha + \beta) \cdot \gamma^*. \tag{4.3}$$

We now prove that

$$(\alpha + \beta) \cdot \gamma^* \subset \alpha + \beta \cdot \gamma^*. \tag{4.4}$$

Since

$$\beta \cdot \gamma^* \subset \alpha + \beta \cdot \gamma^*,$$

we only have to prove that

$$\alpha \cdot \gamma^* \subset \alpha + \beta \cdot \gamma^*. \tag{4.5}$$

Let $p = a \cdot g$ be a point in $\alpha \cdot \gamma^*$ with $a \in \alpha$ and $g \in \gamma^*$. Let b be an arbitrary point in β. Let p_n be a point obtained by the following equation when we regard p_n, a, b, and g as points in the $x-y$ plane:

$$p_n = (1 - 1/n)a + (1/n)(b + n g).$$

Then p_n is on the line segment $\overline{a, (b \cdot g^n)}$, and thus $p_n \in \alpha + \beta \cdot \gamma^*$. Note that $p_\infty = \lim_{n \to \infty} p_n$ is also in $\alpha + \beta \cdot \gamma^*$ and $p_\infty = a \cdot g = p$. Therefore (4.5) holds, and (4.4) does so. □

Lemma 4.5. Let $n \geq 2$. Let α_i, $\gamma_i \in S$ for $i = 1, 2, \ldots, n$. Then

$$\alpha_1 \cdot \gamma_1^* + \alpha_2 \cdot \gamma_2^* + \cdots + \alpha_n \cdot \gamma_n^* = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \cdot \gamma_1^* \cdot \gamma_2^* \cdots \gamma_n^*.$$

Proof. We again use induction on n, and we use k for the index of induction. When $k = 2$, from Lemma 4.4,

$$\alpha_1 \cdot \gamma_1^* + \alpha_2 \cdot \gamma_2^* = (\alpha_1 \cdot \gamma_1^* + \alpha_2) \cdot \gamma_2^* = (\alpha_1 + \alpha_2) \cdot \gamma_1^* \cdot \gamma_2^*.$$

Assume that the lemma holds for $k < n$. Let

$$A = \alpha_1 \cdot \gamma_1^* + \alpha_2 \cdot \gamma_2^* + \cdots + \alpha_n \cdot \gamma_n^*.$$

From the induction hypothesis of $k = n - 1$,

$$A = (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) \cdot \gamma_1^* \cdot \gamma_2^* \cdots \gamma_{n-1}^* + \alpha_n \cdot \gamma_n^*.$$

From Lemma 4.3,

$$A = (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) \cdot (\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1})^* + \alpha_n \cdot \gamma_n^*.$$

From the basis of the induction ($k = 2$),

$$A = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \cdot (\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1})^* \cdot \gamma_n^*.$$

From Lemma 4.3, we have

$$A = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \cdot \gamma_1^* \cdot \gamma_2^* \cdots \gamma_n^*.$$

□

Theorem 4.2. Let

$$\beta_i = \prod_{k=1}^{k_i} \beta_{i,k} = \beta_{i,1} \cdot \beta_{i,2} \cdots \beta_{i,k_i}$$

$$\beta_{i,k} \in \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \text{ for } 1 \le i \le n, \quad \alpha_j \in S.$$

Let $\gamma_i \in S$ for $1 \le i \le n$. Let

$$A = \beta_1 + \beta_2 + \ldots + \beta_n.$$

Let

$$B = \beta_1 \cdot \gamma_1^* + \beta_2 \cdot \gamma_2^* + \ldots + \beta_n \cdot \gamma_n^*.$$

Then

$$| B | \le | A | + 1.$$

Proof. From Lemma 4.3 and 4.5, we have

$$\begin{aligned} B &= \beta_1 \cdot \gamma_1^* + \beta_2 \cdot \gamma_2^* + \cdots + \beta_n \cdot \gamma_n^* \\ &= (\beta_1 + \beta_2 + \cdots + \beta_n) \cdot \gamma_1^* \cdot \gamma_2^* \cdots \gamma_n^* \\ &= (\beta_1 + \beta_2 + \cdots + \beta_n) \cdot (\gamma_1 + \gamma_2 + \cdots + \gamma_n)^*. \end{aligned}$$

From Lemma 4.2,

$$| B | \le | \beta_1 + \beta_2 + \cdots + \beta_n | + 1 = | A | + 1.$$

□

Theorem 4.3. Let $| \alpha | + | \beta | = n$. The operations $\cdot$, $+$ and $*$ can all be done in $O(n \log n)$ steps.

Proof. From Corollary 3.3, we know that the $\cdot$ operation takes $O(n \log n)$ time. In [24, 26], we have an algorithm which takes $O(n \log n)$ time for computing the convex hull of two convex polygons. There is also an algorithm in [26] which computes $l \cap \alpha$ in $O(\log(|\alpha|))$ steps where l is a line segment and α is a convex hull. Therefore the * operation takes $O(n \log n)$ time. □

5. Application of the closed semiring (S, +, · , ∅, ε)

In this section, we define the doubly weighted zero-sum cycle problem and solve it by using the closed semiring defined in Section 2.

Our instance is $I = (G, T)$ where G is a directed graph $G = (V, E)$ and T is a two-dimensional labeling such that $T(e) = (e_x, e_y)$ for every $e \in E$.

Definition. Given a digraph $G = (V, E)$, a *path* P in G is a sequence of vertices $P = v_0, v_1, \ldots, v_k$ where $e_i = (v_{i-1}, v_i) \in E$ and $v_i \in V$. If all vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, a path P is *simple*. A path P such that $v_0 = v_k$ is called a *cycle*. □

Definition. Given a path $P = v_0, v_1, \ldots, v_k$, we define

$$T(P) = \sum_{i=1}^{k} T(e_i) = \left(\sum_{i=1}^{k} e_{i_x}, \sum_{i=1}^{k} e_{i_y} \right)$$

where

$$e_i = (v_{i-1}, v_i) \in E \text{ for } i = 1, 2, \ldots, k \text{ and}$$
$$T(e_i) = (e_{i_x}, e_{i_y}).$$

□

Definition. Let G be a doubly weighted graph. Then a cycle W such that $T(W) = (0, 0)$ is called a *zero-sum cycle*. □

Problem ZSC: Doubly Weighted Zero-sum Cycle Problem.

Instance: The doubly weighted digraph $I = (G, T)$ where G is a directed graph $G = (V, E)$ and T is a two-dimensional labeling such that $T(e) = (e_x, e_y)$ for every $e \in E$.

Question: Does G have a zero-sum cycle? In other words, is there a cycle W such that $T(W) = (0, 0)$?

By using the fact that the two operations defined on convex polygons form a closed semiring, we can answer this question with the Floyd-Warshall's algorithm [10, 11, 12, 14].

Algorithm ZSC:

Input: A doubly weighted graph G is described above. Let $V = \{v_1, v_2, \ldots, v_n\}$. Let T_1 be a function from $V \times V$ to $2^{Z \times Z}$ such that

$$T_1((v, w)) = \begin{cases} \{T((v, w))\} & \text{if } (v, w) \in E \\ \varnothing & \text{otherwise.} \end{cases}$$

Output: This algorithm answers "Yes" if the digraph G has a zero-sum cycle; otherwise the algorithm answers "No".

Method: We compute the convex hull α_{ij}^k for $1 \le i,\ j \le n$ and $0 \le k \le n$. The convex hull α_{ij}^k is the convex hull of the lengths of all paths from v_i to v_j such that all vertices on the path, except possibly the endpoints, are in the set $\{ v_1, v_2, \ldots, v_k \}$.

procedure zero-sum cycle
begin
1. **for** $1 \le i,\ j \le n$ **do** $\alpha_{ij}^0 = T_1 ((v_i, v_j))$;
2. **for** $k = 1$ **to** n
   do
3. **for** $1 \le i,\ j \le n$ **do**
4. $\alpha_{ij}^k = \alpha_{ij}^{k-1} + \alpha_{ik}^{k-1} \cdot (\alpha_{kk}^{k-1})^* \cdot \alpha_{kj}^{k-1}$;
5. **if** $(0, 0) \in \alpha_{ii}^k$ for some i such that $1 \le i \le n$
   then exit ("Yes");
   od
6. exit ("No");
end □

Theorem 5.1. Algorithm ZSC works correctly.

Before proving Theorem 5.1, we need the following lemmas.

Lemma 5.1. If there is a cycle W such that $T(W) = (0, 0)$, there must be some vertex v_i such that $(0, 0) \in \alpha_{ii}^n$.

Proof. Let v_i be a vertex which is on the cycle W. Since the convex hull α_{ii}^n includes all lengths of paths from v_i to v_i, we have $(0, 0) \in \alpha_{ii}^n$. □

Lemma 5.2. If $(0, 0) \in \alpha_{ii}^n$, there must be a cycle W such that $T(W) = (0, 0)$ and the vertex v_i is on the path W.

Proof. Suppose that $(0, 0) \in \alpha_{ii}^n$. Let

$$(\alpha_{ii}^n)_V = \{ s_1, s_2, \ldots, s_m \}$$

such that $s_j \in Z \times Z$. Since α_{ii}^n is a convex polygon, any point $z \in \alpha_{ii}^n$ can be represented as

$$z = \sum_{s_j \in (\alpha_{ii}^n)_V} k_j \, s_j$$

such that $k_j \ge 0$. Let p_j be a cycle corresponding to s_j such that $T(p_j) = s_j$ and v_i is on the cycle p_j. Note that since every s_j has integral coordinates, k_j can be chosen rational, if $z \in Z \times Z$. Thus there are rational numbers k_j such that

$$(0, 0) = \sum_{s_j \in (\alpha_{ii}^n)_V} k_j \, s_j .$$

There is an integer K such that all $K \cdot k_j$ are integers. Thus

$$K \cdot (0, 0) = (0, 0) = \sum_{s_j \in (\alpha_{ii}^n)_V} (K \cdot k_j) s_j.$$

Then the desired cycle W consists of $K \cdot k_j$ copies of p_j for $s_j \in (\alpha_{ii}^n)_V$. □

Now we prove Theorem 5.1.

Proof of Theorem 5.1. From Lemma 5.1 and 5.2, in order to find a zero-sum cycle, we only have to check whether or not there exists some i such that $\{0, 0\} \in \alpha_{ii}^n$. We also have to prove that α_{ij}^k is correctly computed by the algorithm. By induction on k (as in [12]), we can prove that α_{ij}^k is the convex hull of the lengths of all paths from v_i to v_j such that all vertices on the path, except possibly the endpoints, are in the set $\{ v_1, v_2, \ldots, v_k \}$. □

Theorem 5.2. Algorithm ZSC uses $O(n^3)$ $+$, $\cdot$, and $*$ operations from the closed semiring defined above where n is the number of vertices in G.

Proof. Note that line 4 is executed n^3 times in total. □

6. Special cases of the zero-sum problem

In this section, we discuss the special cases of the zero-sum problem where 1) the graphs have one-dimensional labels, 2) the graphs are undirected, 3) the graphs have labels with magnitude at most M, and 4) we are looking for a *simple* cycle with zero sum. The first three cases have low order polynomial algorithms, whereas the fourth is NP-complete.

1) The one-dimensional zero-sum cycle problem

Theorem 6.1. The one-dimensional zero-sum cycle problem can be solved in $O(n^3)$ time where n is the number of vertices.

Proof. We can apply our algorithm ZSC by ignoring the second labels. Note that in the one-dimensional case, every α_{ij}^k has at most two vertices, since it is either a point, a line segment, or a line on the x-axis. Thus $| \alpha_{ij}^k | \le 2$. From Theorem 4.3, each operation $+$, $\cdot$, or $*$ takes constant time. Hence from Theorem 5.2, the algorithm ZSC takes $O(n^3)$ time. □

2) The two-dimensional undirected zero-sum cycle problem

We assume that G is connected. We will show that the undirected version of the zero-sum cycle problem can be solved in $O(m \log m)$ time where m is the number of edges. In the undirected case, a path can traverse an edge in either direction.

An instance of the undirected problem is as follows:

Instance: A connected undirected graph $G = (V, E)$ with

$$V = \{ v_1, v_2, \ldots, v_n \} \text{ and } E = \{ e_1, e_2, \ldots, e_m \}.$$

A two-dimensional labeling T from E to $Z \times Z$ with $T(e) = (e_x, e_y)$ for every $e \in E$.

Now we have the following lemma:

Lemma 6.1. Let G and T be defined as above. Let H_G be the convex hull of $\{ T(e) \mid e \in E \}$. A necessary and sufficient condition for the existence of a zero-sum cycle is that exactly one of the following two conditions holds:

1) The convex hull H_G properly contains the origin.
2) The origin is on an edge h of the convex polygon H_G. Let $Y = \{ e \in E \mid T(e) \textit{ is on } h \}$. Then there exists an edge $e \in Y$ such that $T(e) = (0, 0)$, or there are two edges $e_1, e_2 \in Y$ such that e_1 and e_2 are adjacent in G and the origin is on the line segment $\overline{T(e_1), T(e_2)}$.

Before proving Lemma 6.1, we need some definitions. Let

$$X = \{ C_e \mid e \in E \}$$

such that C_e is the cycle $v \to w \to v$ where $e = (v, w)$. Then

$$T(C_e) = 2T(e).$$

Definition. A set of cycles $A = \{ W_i \mid i \in I \}$ is said to be *nullable* if there exists a set of non-negative integers $A_Z = \{ n_i \in Z^+ \cup \{0\} \mid i \in I \}$ such that the n_i are not all 0 and $\sum_{i \in I} n_i T(W_i) = (0, 0)$. If $(\bigcup_{i \in I} W_i)$ is connected, we say that A is *connected.* □

Note that we can construct a zero-sum cycle from a connected nullable set. Now we have the following lemmas.

Lemma 6.2. Let G, T, and X be defined as above. Let $A = \{ W_i \mid i \in I \}$ be a nullable set of cycles. Then we can find a connected nullable set B.

Proof. Since A is nullable, there exists a set of non-negative integers $A_Z = \{ n_i \in Z^+ \cup \{0\} \mid i \in I \}$ such that the n_i are not all 0 and $\sum_{i \in I} n_i T(W_i) = (0, 0)$. If A is connected, A is the desired set. Suppose A is not connected. Let v_i be an arbitrary point on W_i for every $i \in I$. Since G is connected, there is a cycle P_i which passes through v_1 and v_i for every $i \in I - \{1\}$. Let k be a large positive integer. Let Q_i be a cycle consisting of k copies of W_i and one copy of P_i for every $i \in I - \{1\}$. Let $Q_1 = W_1$. Then

$$T(Q_i) = \begin{cases} kT(W_1) & \text{for } i = 1, \\ kT(W_i) + T(P_i) & \text{for } i \in I - \{1\}. \end{cases}$$

Since the convex hull of $\{ T(W_i) \mid i \in I \}$ contains $(0, 0)$, the convex hull of $\{ T(Q_i) \mid i \in I \}$ contains $(0, 0)$ for some large k. Therefore $B = \{ Q_i \mid i \in I \}$ is nullable for large k. Since $v_1 \in \bigcap_{i \in I} Q_i$, B is connected. Thus B is the desired set.

□

Now we prove Lemma 6.1.

Proof of Lemma 6.1. Suppose 1) holds. Note that $T(C_e) = 2T(e)$ where C_e is the cycle for every $e \in E$ defined as above. Since $A = \{ 2T(e) \mid e \in E \}$ is a nullable set, we can find a connected nullable set, by Lemma 6.2. Thus there is a zero-sum cycle in G. Suppose 2) holds. It is obvious that there is a zero-sum cycle in G.

Conversely, suppose there exists a zero-sum cycle W. From the definition, there exist positive integers n_e for $e \in W$ such that

$$\sum_{e \in W} n_e T(e) = (0, 0). \tag{6.1}$$

This means that the convex hull of $\{ T(e) \mid e \in E \}$, denoted by H_G, contains the origin. If H_G contains the origin properly, 1) holds. Otherwise, there must be an edge $e \in E$ such that $T(e) = (0, 0)$, or the origin must be on an edge h of H_G. Now we assume that $T(e) \neq (0, 0)$ for every $e \in E$. Let

$$Y = \{ e \in E \mid T(e) \textit{ is on the edge } h \}.$$

Since W is nullable, every edge in W is in Y. Let $\bar{e}$ be an edge in Y. Then for every edge $e \in Y$, there exists k_e and $T(e) = k_e T(\bar{e})$. Let

$$W_+ = \{ e \in W \mid T(e) = k_e T(\bar{e}), k_e > 0 \},$$

and let

$$W_- = \{ e \in W \mid T(e) = -k_e T(\bar{e}), k_e > 0 \}.$$

From (6.1), we have

$$\sum_{e \in W} n_e T(e) = \left(\sum_{e \in W_+} n_e k_e - \sum_{e \in W_-} n_e k_e \right) T(\bar{e}) = (0, 0). \tag{6.2}$$

Note that $W_+ \neq \varnothing$ and $W_- \neq \varnothing$. From (6.2), since $W = W_+ \cup W_-$ is connected, there must be connected edges $e_1 \in W_+$ and $e_2 \in W_-$. Thus 2) holds. □

Theorem 6.2. The two-dimensional undirected zero-sum cycle problem can be solved in $O(m \log m)$ time where m is the number of edges.

Proof. We only have to check condition 1) and 2) in Lemma 6.1, which can be done in $O(m \log m)$ time. □

3) The graphs with bounded labels

Definition. Let $G = (V, E)$ be a digraph. Let T be a two-dimensional labeling from E to $Z \times Z$. Let M be a positive integer. An instance (G, T) is called a *M-bounded graph* if each dimension of every label is in $[-M, M]$; that is,

$$| e_x | \leq M \text{ and } | e_y | \leq M \text{ for every } e \in E,\ T(e) = (e_x, e_y).$$

□

In many VLSI applications, the communication between regular cells is made locally: that is, interconnections are made only to neighbors. For example, the $n \times n$ multiplier can be constructed from arrays of one-bit full adders with carry and sum signal connections to the neighbors of each cell [1 - 4]. The parallel adder can also be constructed from one-bit full adders with the carry connection to the neighbor of each cell [1]. Many systolic arrays are also implemented with interconnections to neighbors. In such VLSI applications, the associated static digraphs of the regular structure are all 1-bounded graphs [4].

We have the following lemma about the number of edges of a convex polygon included in a bounded region.

Lemma 6.3. Let R be a rectangle of width w and height h. Let H be an arbitrary convex polygon included in R. Then $| H | \leq 2 \max (w, h) + 2$.

Proof. Without loss of generality, we can assume that $\max (w, h) = w$. Let H_u be the set of edges in H from its leftmost and uppermost vertex to its rightmost and uppermost vertex in clockwise order. When we traverse an edge in H_u, we move at least one unit in the x-direction. Thus the number of edges in H_u is at most w. There are at most two vertical edges in H. Thus

$$| H | \leq 2 \max (w, h) + 2.$$

□

Lemma 6.4. Let G be a M-bounded graph with n vertices, then we have

$$| \alpha_{ij}^k | \leq 4nM + 3.$$

Proof. Let β_{ij}^k be the convex hull of the lengths of all simple paths from v_i to v_j such that all vertices on the path, except possibly the endpoints, are in the set $\{ v_1, v_2, \ldots, v_k \}$. Note that the length of a simple path is at most nM in each dimension. Thus β_{ij}^k is bounded by the rectangle $[-nM, nM] \times [-nM, nM]$. Therefore, from Lemma 6.3,

$$| \beta_{ij}^k | \leq 2 \cdot (2nM) + 2 = 4nM + 2.$$

From Theorem 4.2,

$$| \alpha_{ij}^k | \leq | \beta_{ij}^k | + 1 \leq 4nM + 3.$$

□

Theorem 6.3. The algorithm ZSC takes $O(n^4 M \log(nM))$ time for the M-bounded graphs with n vertices.

Proof. From Theorem 5.2 and Lemma 6.4, the algorithm ZSC takes $O(n^3 \cdot nM \log(nM)) = O(n^4 M \log(nM))$ time. □

4) The zero-sum simple cycle problem

Theorem 6.4. The zero-sum simple cycle problem (ZSSC) is NP-complete.

Proof. Here we use a variant of the reduction from the subset sum to the directed path problem in the one-dimensional dynamic graphs discussed in [13]. It is obvious that ZSSC is in NP. We use reduction from the subset sum problem SS to ZSSC, where the problem SS is defined as follows:

Input: $\{ a_i \in Z^+ \mid i \in I \}$ where $I = \{ 1, 2, \ldots, n \}$ and $B \in Z^+$.
Question: Is there a subset J of I such that $\sum_{j \in J} a_j = B$?

Given an instance I_{SS} of SS, we construct an instance I_{ZSSC} of the zero-sum simple cycle problem as follows: A directed graph $G = (V, E)$ is shown in Fig. 6 where

$$\begin{aligned} V &= \{ v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n \}, \\ E &= \{ e_i = (v_{i-1}, v_i) \mid i = 1, 2, \ldots, n \} \cup \\ &\quad \{ f_i = (v_{i-1}, w_i) \mid i = 1, 2, \ldots, n \} \cup \\ &\quad \{ g_i = (w_i, v_i) \mid i = 1, 2, \ldots, n \} \cup \\ &\quad \{ e_0 = (v_n, v_1) \}. \end{aligned}$$

Let T be a two-dimensional labeling from E to $Z \times Z$ as follows:

$$\begin{cases} T(e_0) = (-B, 0), & \\ T(e_i) = T(g_i) = (0, 0) & \text{for } i = 1, 2, \ldots, n, \\ T(f_i) = (a_i, 0) & \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Suppose I_{SS} has a solution J such that $\sum_{j \in J} a_j = B$. Then I_{ZSSC} has a solution of a simple cycle consisting of e_0, f_j and g_j for $j \in J$, and e_i for $i \in \bar{J}$.

Conversely, suppose that I_{ZSSC} has a solution; that is, there exists a simple cycle W such that $T(W) = (0, 0)$. Note that W must use e_1. Let $J = \{ j \mid f_j \in W \}$. Then $\sum_{j \in J} a_j = B$. Thus I_{SS} has the solution J. □

7. Backedged two-terminal series-parallel multidigraphs.

The class of Two-Terminal Series-Parallel (TTSP) graphs has been well studied, the undirected version in [16, 17, 18, 19] from the aspect of its relationship to electrical networks, and the directed version in [15] to provide an algorithm to recognize general series-parallel digraphs.

A digraph is called a *multidigraph* if we allow multiple edges between the same two vertices. The definition of the class of TTSP multidigraphs appears in [15] as follows:

Definition. [Two-Terminal Series-Parallel Multidigraphs].

(1) A digraph consisting of two vertices joined by a single edge is in TTSP.

(2) If G_1 and G_2 are TTSP multidigraphs, so too is the multidigraph obtained by either of the following operations:

(2.a) *Two terminal parallel composition:* identify the source of G_1 with the source of G_2 and the sink of G_1 with the the sink of G_2.

(2.b) *Two terminal series composition:* identify the sink of G_1 with the source of G_2.

Let $TTSP(\, m\,)$ be the class of TTSP multidigraphs which have m edges. □

From this definition, a TTSP multidigraph has the single source, denoted by s, and the single sink, denoted by t. We define the more general class called BTTSP as follows:

Definition. [Backedged Two-Terminal Series-Parallel Multidigraph].

Let G be a TTSP graph. A multidigraph G_B, obtained by adding any number of *backedges* to a TTSP graph G, is called a *BTTSP* (*Backedged Two-Terminal Series-Parallel*) multidigraph. An edge (x, y) is called a *backedge* if there is a path from y to x in G. The graph G is called *the underlying TTSP graph of* G_B.

Let $BTTSP(\, m\,)$ be the class of BTTSP multidigraphs which have m edges. □

Fig. 7a shows an example of a BTTSP graph G_B which consists of a backedge indicated by dotted lines and the underlying TTSP graph G.

Let $G = (\, V,\, E\,)$ be a multidigraph. Let T be a two-dimensional labeling from E to $Z \times Z$. Then for all $x,\, y \in V$, *the convex polygon of* $x - y$ *paths*, $\alpha_{xy}(\, T\,)$, is defined in the same way as in previous sections; that is, the convex polygon $\alpha_{xy}(\, T\,)$ is the convex hull of all lengths of paths from x to y in G with the two-dimensional labeling T.

Definition. For any multidigraph G, let

$$A(\, G\,) = \max_{x,\, y,\, T} \mid \alpha_{xy}(\, T\,) \mid$$

and similarly for a class of graphs we write $A(\{G\})$. That is, $A(G)$ is the maximum number of edges in $\alpha_{xy}(T)$ when x, y, and T are arbitrary and G is fixed. □

Definition. Let L_m be the TTSP multidigraph consisting of two vertices, s and t, and m edges from s to t. □

See Fig. 7c for an example.

We have the following theorem:

Theorem 7.1. $A(TTSP(m)) = m$.

Before proving the theorem, we need some lemmas:

Lemma 7.1. $A(L_m) = m$.

Proof. Let e_i for $i = 1, 2, \ldots, m$ be the edges of L_m, and let $T(e_i) = v_i \in Z \times Z$. Then α_{st} is the convex hull of $\{v_i\}$, which can clearly have m sides, and no more than m sides. □

Lemma 7.2. Let G be in $TTSP(m)$ with source s and sink t, and let T be a two-dimensional labeling of G. Let x, y be arbitrary vertices in G such that $(x, y) \neq (s, t)$. Then there exists a two-dimensional labeling T' such that

$$|\alpha_{xy}(T)| \leq |\alpha_{st}(T')|.$$

Before proving Lemma 7.2, we define the graph G_{xy} as follows:

Definition. Let $G = (V, E)$ be a TTSP graph. Let $x, y \in V$. Then we define the graph $G_{xy} = (V_{xy}, E_{xy})$ by the following operations on G:

1) First, we delete all incoming edges to x and all outgoing edges from y.
2) We then delete all *useless* vertices and their adjacent edges. A vertex v is called *useless* when there is no $v-x$ path or $y-v$ path. □

Proof of Lemma 7.2. If there is no $x-y$ path in G, we have $\alpha_{xy}(T) = \emptyset$. Thus

$$|\alpha_{xy}(T)| = 0 \leq |\alpha_{st}(T)|.$$

Choose T as T'.

Otherwise there exists an $x-y$ path in G. Since there exist an $s-x$ path and a $y-t$ path, let P_{sx} (P_{yt}) be an arbitrary $s-x$ path ($y-t$ path). Let $G_1 = (V_1, E_1)$ be the graph consisting of P_{sx}, G_{xy}, and P_{yt}. We define a two-dimensional labeling T' as follows:

$$T'(e) = \begin{cases} \emptyset & \text{if } e \in E - E_1 \\ \varepsilon & \text{if } e \in P_{sx} \cup P_{yt}. \\ T(e) & \text{if } e \in E_{xy} \end{cases}$$

Then

$$| \alpha_{xy}(T) | = | \alpha_{st}(T') |.$$

□

Proof of Theorem 7.1. We first prove $A(TTSP(m)) \leq m$ by induction on m. It is clear that $A(TTSP(1)) = 1$. Assume that the induction hypothesis is true for $k < m$. Let $G = (V, E)$ be in $TTSP(m)$ with source s and sink t. From Lemma 7.2, we only have to show $| \alpha_{st}(T) | \leq m$ for any T. From the definition of TTSP, G must be constructed either in series or in parallel from $G_1 \in TTSP(m_1)$ and $G_2 \in TTSP(m_2)$ such that $m = m_1 + m_2$ and $m_1, m_2 > 0$. Then we have

$$\begin{aligned} A(G) &\leq A(G_1) + A(G_2) \quad \text{(from Theorems 3.2 and 4.1)} \\ &\leq m_1 + m_2 = m. \quad \text{(from the induction hypothesis)} \end{aligned}$$

Thus

$$A(TTSP(m)) \leq m.$$

Since $L_m \in TTSP(m)$, from Lemma 7.1, $A(L_m) = m$, which shows this bound is achievable. □

We will show the same result for the class of BTTSP multidigraphs. The following lemma says that every backedge in an $s - t$ path in a BTTSP graph lies on a cycle which lies on the $s - t$ path.

Lemma 7.3. Let G_B be a BTTSP graph with source s and sink t, and let P be a path from s to t possibly using some backedges in G_B. Then P can be represented as follows: $P = P_1 C_1^{r_1} P_2 C_2^{r_2} \ldots P_k C_k^{r_k}$ where $P_1 P_2 \ldots P_k$ is a path from source to sink in the underlying TTSP graph G, the C_i's are cycles in G_B, and $r_i \geq 0$ for $1 \leq i \leq k$.

Proof. See next section. □

Theorem 7.2. $A(BTTSP(m)) = m$.

Proof. Since $TTSP(m) \subset BTTSP(m)$, we have

$$A(BTTSP(m)) \geq A(TTSP(m)) = m.$$

We now prove that for an arbitrary graph $G_B \in BTTSP(m)$ with at least one backedge, $A(G_B) \leq m$. Let $G = (V_1, E_1)$ be the underlying TTSP graph of G_B, and let T be a two-dimensional labeling of G_B. Let $P_B(s, t)$ be the set of $s - t$ paths in G_B, and let $P(s, t)$ be the set of $s - t$ paths in G. Let P be an arbitrary path in $P_B(s, t)$. Then from Lemma 7.3, P can be represented as

$$P = P_1 C_1^{r_1} P_2 C_2^{r_2} \ldots P_k C_k^{r_k}$$

where $P_1 P_2 \ldots P_k$ is a path from source to sink in the underlying TTSP graph G, the C_i's are cycles in G_B, and $r_i \geq 0$ for $1 \leq i \leq k$. Let

$$\beta_P = T(P_1 P_2 \ldots P_k) \text{ and } \gamma_{P_i} = T(C_i) \text{ for } 1 \leq i \leq k.$$

Then

$$T(P) = \beta_P \cdot \gamma_{P_1}^{r_1} \cdot \gamma_{P_2}^{r_2} \cdots \gamma_{P_k}^{r_k}.$$

Let

$$\begin{aligned} P^* = \{ P_1 C_1^{n_1 r_1} P_2 C_2^{n_2 r_2} \ldots P_k C_k^{n_k r_k} \mid \\ P = P_1 C_1^{r_1} P_2 C_2^{r_2} \ldots P_k C_k^{r_k} \in P_B(s, t), \\ \text{and } n_i \in Z^+ \cup \{0\} \text{ for } 1 \leq i \leq k\}. \end{aligned}$$

Let $T(P^*)$ be defined as $T(P^*) = \sum_{Q \in P^*} T(Q)$. Since $P^* \subset P_B(s, t)$, we have

$$T(P^*) = \beta_P \cdot (\gamma_{P_1} + \gamma_{P_2} + \cdots + \gamma_{P_k})^* \subset \sum_{P \in P_B(s, t)} T(P).$$

Note that $T(P)$ is in $T(P^*)$. Therefore

$$\sum_{P \in P_B(s, t)} T(P) = \sum_{P \in P_B(s, t)} T(P^*).$$

Thus we now have

$$\begin{aligned} |\alpha_{st}(T)| &= \Big|\sum_{P \in P_B(s, t)} T(P)\Big| = \Big|\sum_{P \in P_B(s, t)} T(P^*)\Big| \\ &= \Big|\sum_{P \in P_B(s, t)} \beta_P \cdot (\gamma_{P_1} + \gamma_{P_2} + \cdots + \gamma_{P_k})^*\Big| \\ &\leq \Big|\sum_{P \in P_B(s, t)} \beta_P\Big| + 1 \quad (\text{ using Theorem 4.2 }) \\ &\leq \Big|\sum_{P \in P(s, t)} T(P)\Big| + 1 \\ &= |\alpha_{st}(T)| + 1 \quad (\text{ from the definition }) \\ &\leq A(TTSP(|E_1|)) + 1 \\ &= |E_1| + 1 \quad (\text{ using Theorem 7.1 }) \\ &\leq m \end{aligned}$$

because $|E_1| \leq m - 1$ by the assumption that G has a backedge. Thus

$$A(BTTSP(m)) \leq m.$$

□

Corollary 7.2. For BTTSP, the algorithm ZSC runs in O(n^3 m log m) time where m is the number of vertices and the e is the number of edges.

Proof. Clear from Theorem 5.2 and Theorem 7.2. □

8. Proof of Lemma 7.3.

Let G = (V, E) be a TTSP multidigraph with source s and sink t. A *binary decomposition tree for the TTSP*, which was discussed in [15], represents the construction process of the TTSP by a binary tree. A binary decomposition tree, called BDT(G), has a leaf for each of the edges of the TTSP multidigraph it represents. Fig. 7b shows an example, where ${}_vS_w$ (${}_vP_w$) indicates the two-terminal series (parallel) composition with source v and sink w. Note that every path in G has a corresponding route in BDT(G). For example, the path

$$P = v - 3 - b - 5 - c - 7 - d - 8 - e - 9 - w,$$

shown in bold lines in Fig. 7a, has the corresponding route in BDT(G)

$$P_{BDT} : (\mathbf{v}, \mathbf{b}) = 3 - {}_vP_{\mathbf{b}} - {}_aS_{\mathbf{b}} - {}_aS_{\mathbf{c}} - {}_aP_{\mathbf{c}} - {}_sS_{\mathbf{c}} - {}_sS_{\mathbf{d}} -$$

$$ {}_sS_t - {}_{\mathbf{d}}S_t - {}_{\mathbf{d}}P_g - {}_{\mathbf{d}}S_g - {}_{\mathbf{d}}S_f - {}_{\mathbf{e}}P_f - {}_{\mathbf{e}}S_f - 9 = (\mathbf{e}, \mathbf{w}).$$

Note that the vertices shown in bold face in the path P_{BDT}, (b, c, d, and e), appear in P in this order.

Let T_{vw} be the smallest subtree in BDT(G) which includes vertices v and w. (Find the first common ancestor, and include the appropriate subtree.) Let T_v (T_w) be the subtree of T_{vw} in which v (w) exists as shown in Fig. 7d. We use ${}_aA_b$ for representing either ${}_aS_b$ or ${}_aP_b$. Let ${}_aA_b$, ${}_aA_c$, and ${}_dA_b$ be the root of the subtrees T_{vw}, T_v, and T_w, respectively. Then we have the following lemma:

Lemma 8.1. Suppose ${}_xA_y$ appears in P_{BDT}.

If ${}_xA_y$ appears in T_v, then y is in P.

If ${}_xA_y$ appears in T_w, then x is in P.

Proof. Suppose ${}_xA_y$ appears in T_v. The vertex v is in the TTSP graph with source x and sink y. Thus every path from v to a vertex which is not in T_v must pass through y. We can prove the other case in the same way. □

Corollary 8.1. Suppose there is a $v-w$ path in G and T_v, T_w, and T_{vw} are defined as above. Let ${}_aA_b$, ${}_aA_c$, and ${}_dA_b$ be the roots of the subtrees T_{vw}, T_v, and T_w, respectively. Then we have the following:

1) ${}_aA_b = {}_aS_b$, that is, the root of T_{vw} corresponds to a series composition, and $c = d$.

2) Every path from v to t passes through the vertex c.

3) Every path from s to w passes through the vertex c.

4) Any $v-t$ path and any $s-w$ path intersect at some vertex.

Proof. 1) If the root of T_{vw} corresponds to a parallel composition, there is no path from v to w. Thus ${}_aA_b = {}_aS_b$. And the series composition identifies the sink of ${}_aA_c$ and the source of ${}_dA_b$, thus $c = d$.

2) Since there is a $w-t$ path and $t \in \overline{T_v}$. From the proof of the above lemma, every path from v to t passes through the vertex c.

3) We can prove this in the same way as 2).

4) It is obvious from 3) and 4). □

Proof of Lemma 7.3. Let k be the number of backedges in P. Let B_{xy} (P_{xy}) denote an $x-y$ path in G_B (G). We prove the lemma by induction on k.

Suppose $k = 1$ and let $e = (w, v)$ be the backedge in P. Note that there must be a $v-w$ path in the underlying TTSP graph G. P can be represented as

$$P = P_{sw}\ e\ P_{vt}.$$

P_{sw} and P_{vt} are paths in G, since $k = 1$, so that from Corollary 8.1, they pass through the same vertex c. Therefore, we can express P as

$$P = P_{sc}\ P_{cw}\ e\ P_{vc}\ P_{ct}.$$

Thus we obtain the cycle

$$C_1 = P_{cw}\ e\ P_{vc}.$$

Suppose the lemma holds for numbers less than k. Let $E_B = \{ e_1, e_2, \ldots, e_l \}$ be the backedges which appear in P, in this order. Let $e_i = (w_i, v_i)$ for $1 \le i \le l$. Let $e_f = (w_f, v_f)$ be the last backedge in E_B such that there is a path from v_f to w_1 in G. Assume that $e_f \ne e_1$. (When $e_f = e_1$, we can easily modify the following proof.) Then as shown in Fig. 7e, P can be represented as

$$P = P_{sw_1} e_1 B_{v_1 w_f} e_f B_{v_f t}.$$

Let P_{sv_f} be an arbitrary $s-v_f$ path in G, and let

$$P_1 = P_{sv_f} B_{v_f t}.$$

Then P_1 has l backedges where $l \le k - 1$, because e_1 is not on P_1. From the induction hypothesis, P_1 can be formed from an $s-t$ path P_{st} and cycles $\{ C_j \mid j \in J \}$. Note that P_{sv_f} is part of P_{st}; that is, there exists a v_f-t path $P_{v_f t}$ such that $P_{st} = P_{sv_f} P_{v_f t}$. Suppose not. Let $e = (x, y)$ be the first edge in P_{sv_f} such that $y \in \overline{P_{st}}$. Then there exists a backedge (z, x) in $B_{v_f t}$ and a cycle C_j such that $(z, x) \in C_j$. Since there exists an $x-v_f$ path and a v_f-w_1 path in G, there exists a $x - w_1$ path in G. This contradicts the definition of v_f, since the backedge (z, x) is in $B_{v_f t}$, and thus appears after e_f in E_B.

Thus $P_{st} = P_{sv_f} P_{v_f t}$, and the path P consists of $P_{sw_1} e_1 B_{v_1 w_f} e_f$, $P_{v_f t}$, and cycles $\{ C_j \mid j \in J \}$. Let

$$P_2 = P_{sw_1} e_1 B_{v_1 w_f} e_f P_{v_f t}.$$

Since there is a $v_f - w_1$ path in G, from Corollary 8.1, P_{sw_1} and $P_{v_f t}$ intersect at some vertex c. Thus P_2 can be expressed as

$$P_2 = P_{sc} \tilde{C} P_{ct} \text{ where } \tilde{C} = P_{cw_1} e_1 B_{v_1 w_f} e_f P_{v_f c}.$$

Therefore the path P consists of the path $P_{sc} P_{ct}$, cycle $\tilde{C}$ and cycles $\{ C_j \mid j \in J \}$.
□

9. Conclusion.

We showed that the two operations of vector summation (·) and convex hull of union (+) defined on the set of convex polygons form a closed semiring. We then investigated some properties of these operations. For example, the · operation can be done in $O(m \log m)$ time where m is the number of edges involved in the operation, and the decomposability problem, which can be regarded as the inverse operation of the · operation, is shown to be NP-complete.

We then obtained the algorithm ZSC by using Kleene's closure algorithm on the above closed semiring. The algorithm ZSC solves the two-dimensional zero-sum cycle problem, which has a close relationship to the problem of acyclicity in two-dimensional regular electrical circuits. The complexities of our algorithm ZSC in some special cases are $O(n^3)$ time for the one-dimensional labeling case, $O(n^4 M \log (nM))$ time for M-bounded graphs, and $O(n^3 m \log m)$ time for BTTSP graphs, where n is the number of vertices and m is the number of edges. The complexity of this algorithm in the general case remains open. We also showed that the undirected version of the zero-sum cycle problem can be solved in $O(m \log m)$ time, and that the zero-sum *simple* cycle problem is NP-complete.

We are now working on the following conjecture about the number of edges of the convex polygons which appear in the algorithm ZSC:

Conjecture. Let G, T, and $\alpha_{ij}(T)$ be defined in the same way as in the text. Then

$$A(G) = \max_{i,\, j,\, T} \mid \alpha_{ij}(T) \mid \leq m$$

where m is the number of edges in G.

If this conjecture is true, then algorithm ZSC runs in $O(n^3 m \log m)$ time on general graphs.

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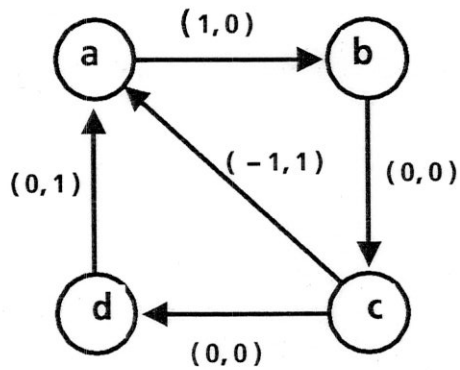
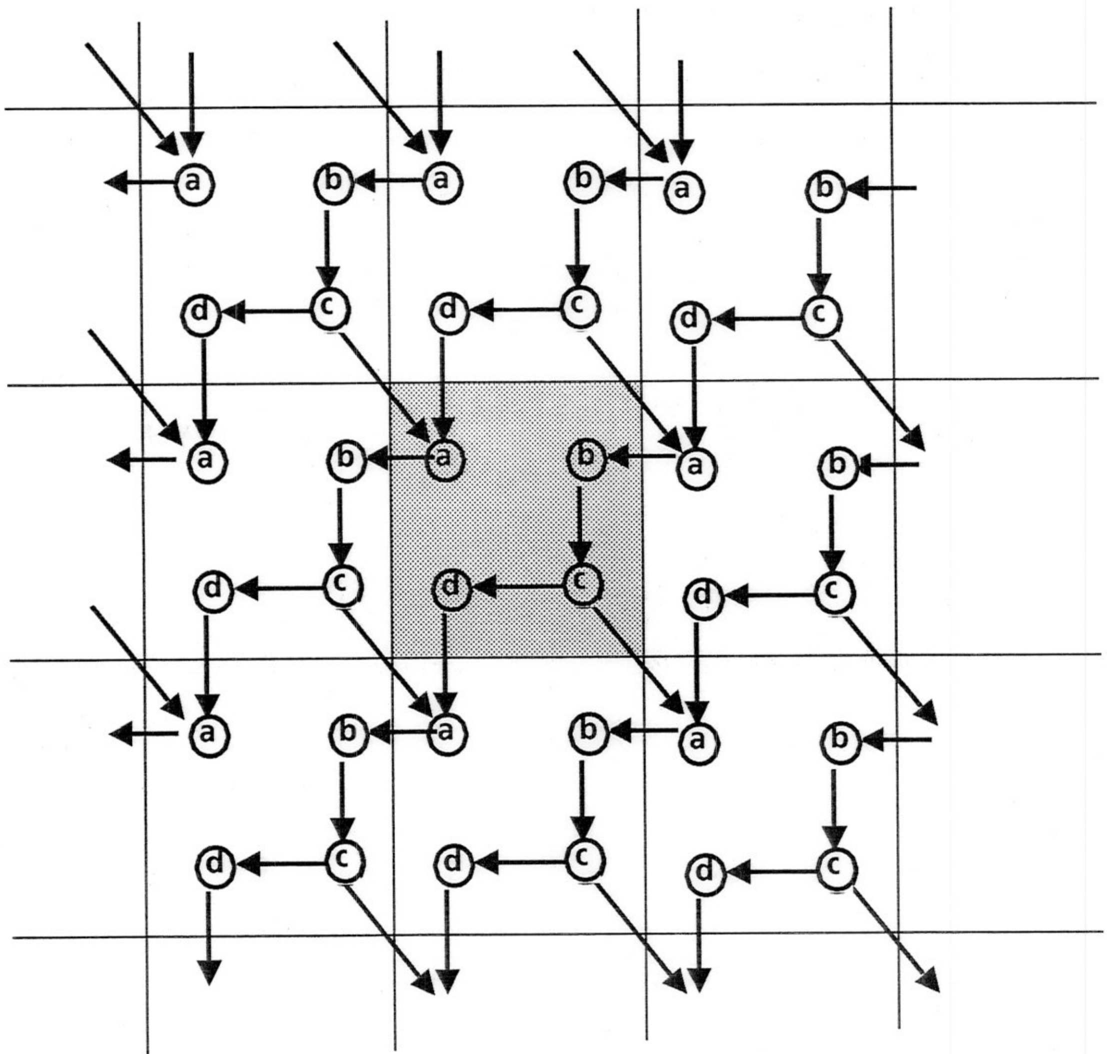


A static graph G



The dynamic graph $G^{\infty,\infty}$

Figure 1. A static graph G shows how to connect nodes in $G^{\infty,\infty}$.
The shaded area shows a basic cell.

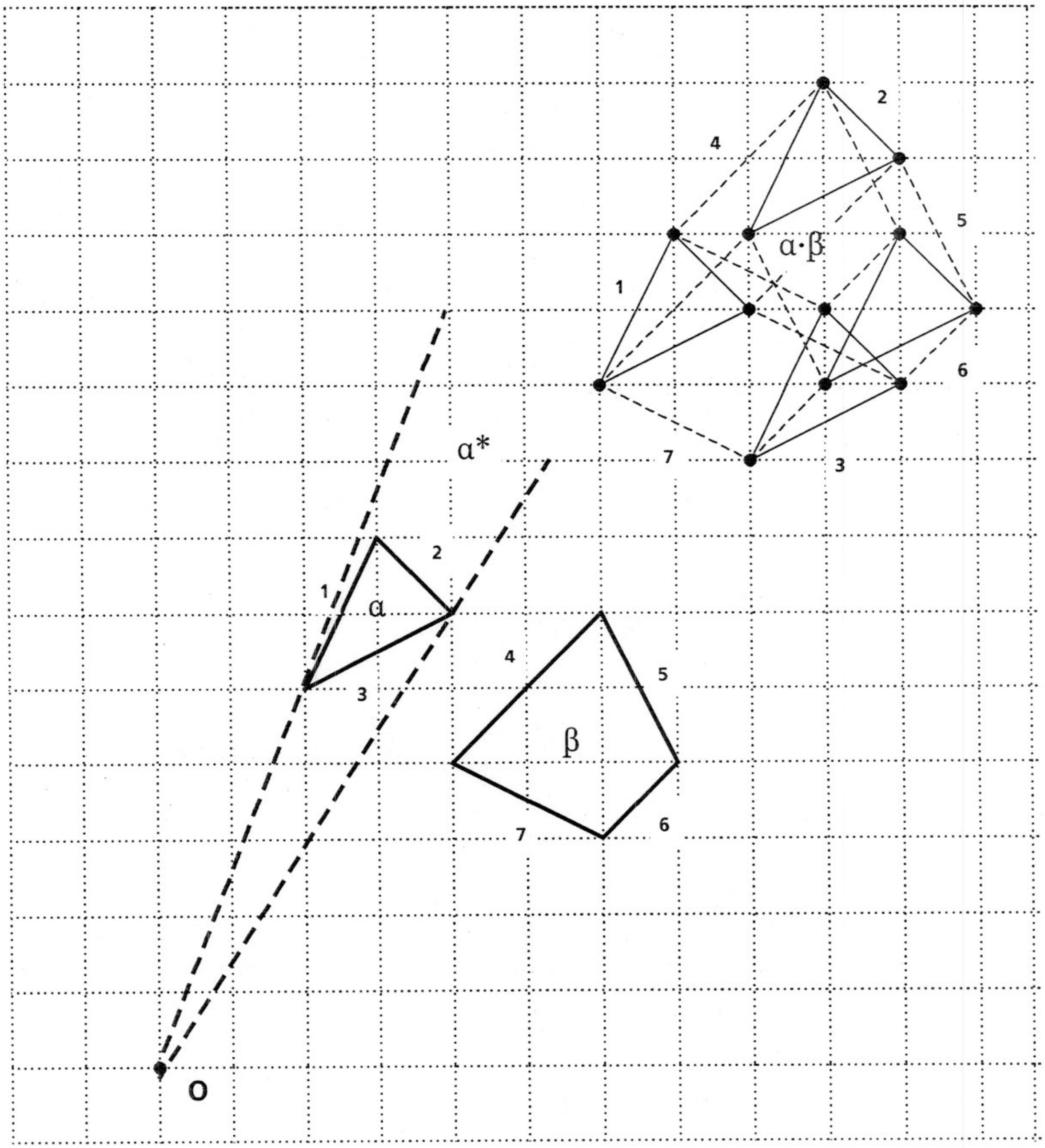


Figure 2. α·β is bounded by edges which are equivalent to the edges in α or β. The equivalent edges are shown by the same numbers.

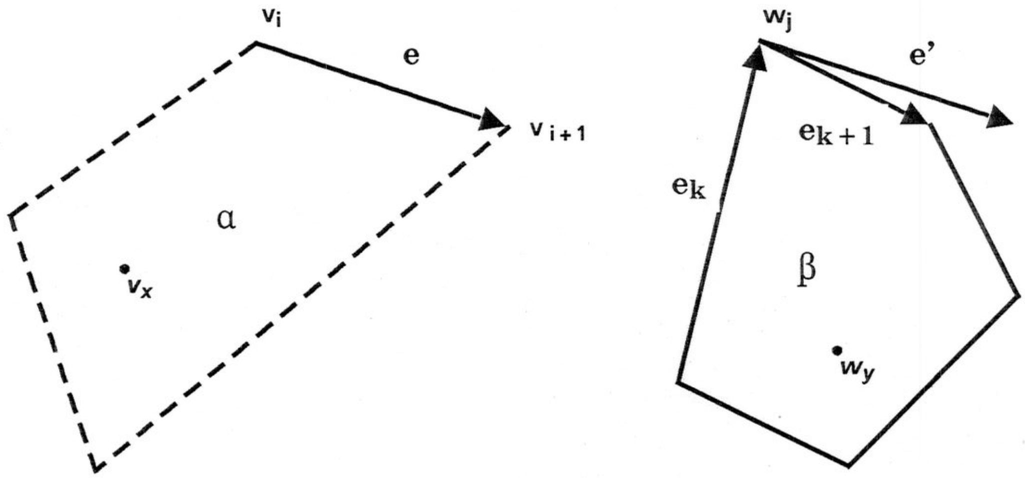


Figure 3a. There exists k such that θ_e is between θ_{e_k} and $\theta_{e_{k+1}}$ in clockwise order.

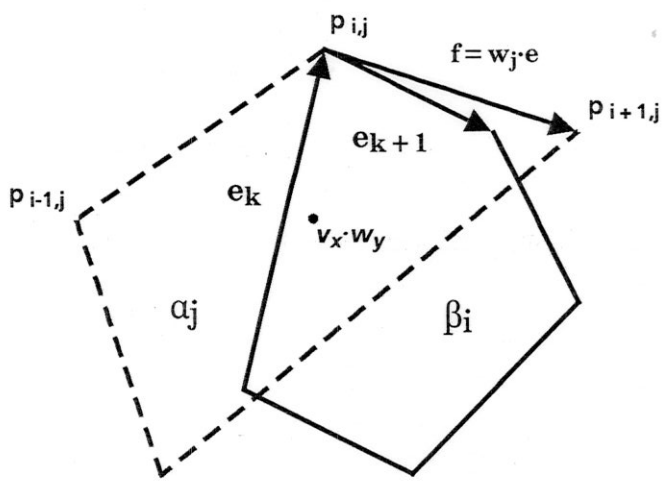


Figure 3b. Every point $v_x{\cdot}w_y$ in $\alpha{\cdot}\beta$ is in the right hand side of $f = w_j{\cdot}e$ where $v_x \in \alpha$ and $w_y \in \beta$.

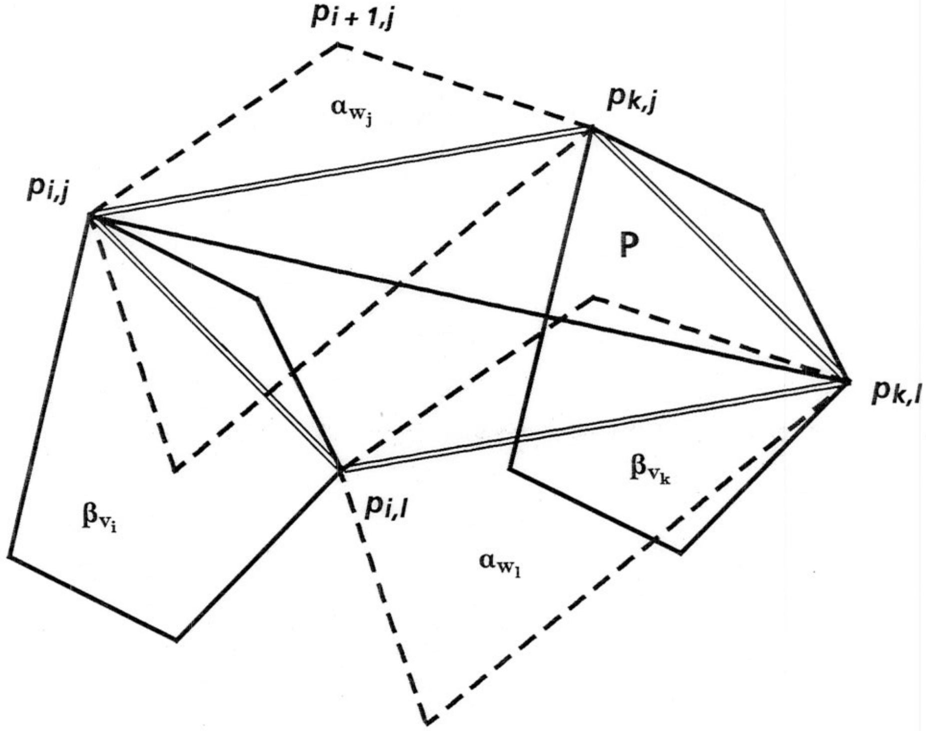


Figure 3c. **This shows part of $\alpha \cdot \beta$. A line segment $p_{i,j}, p_{k,l}$ cannot be an edge of $\alpha \cdot \beta$, when $|k-i| + |l-j| \geq 2$.**

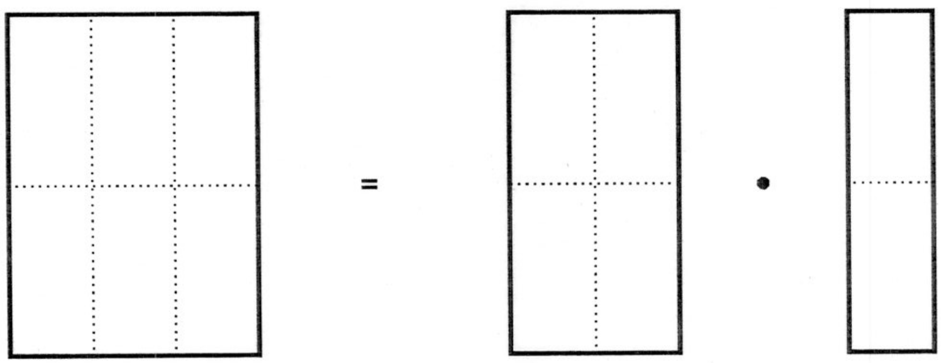


Figure 4a. Example of an invalid decomposition into non-independent convex polygons.

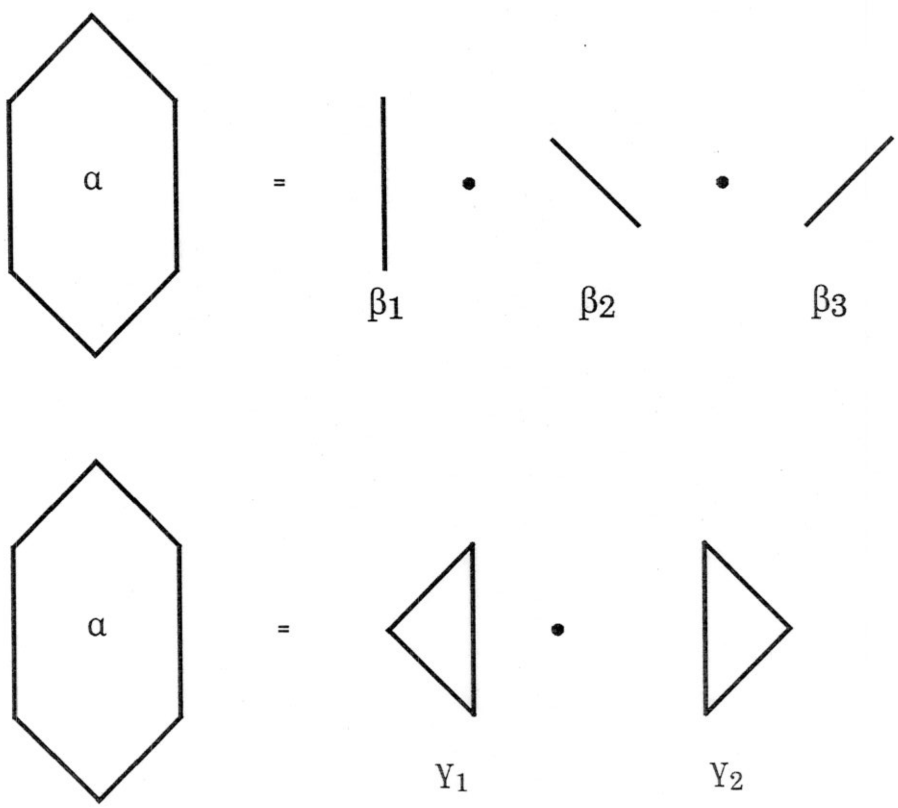


Figure 4b. α **is decomposed in two different ways.**

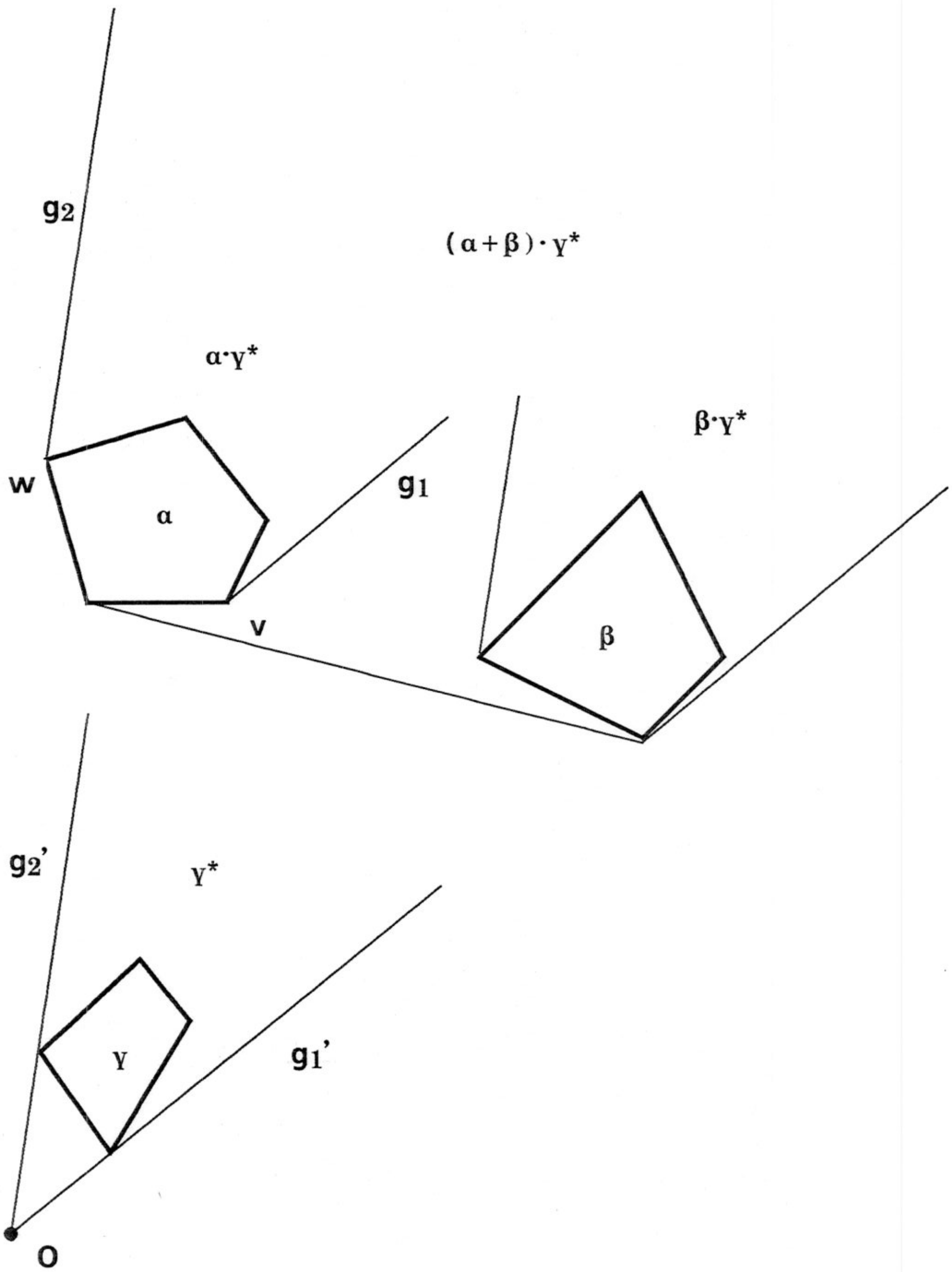


Figure 5. $|\alpha \cdot \gamma^*| \leq |\alpha| + 1$ and $\alpha + \beta \cdot \gamma^* = (\alpha + \beta) \cdot \gamma^*$.

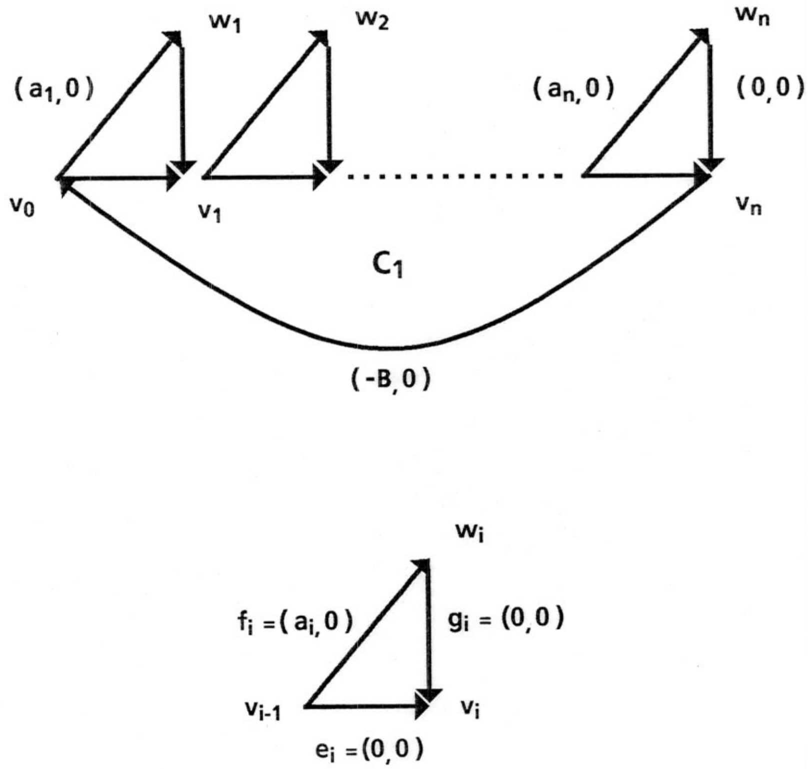


Figure 6. The graph above has a zero-sum cycle if and only if there exists a subindex $J \subset I = \{1, 2, ..., n\}$ such that $\Sigma_{j \in J} a_j = B$.

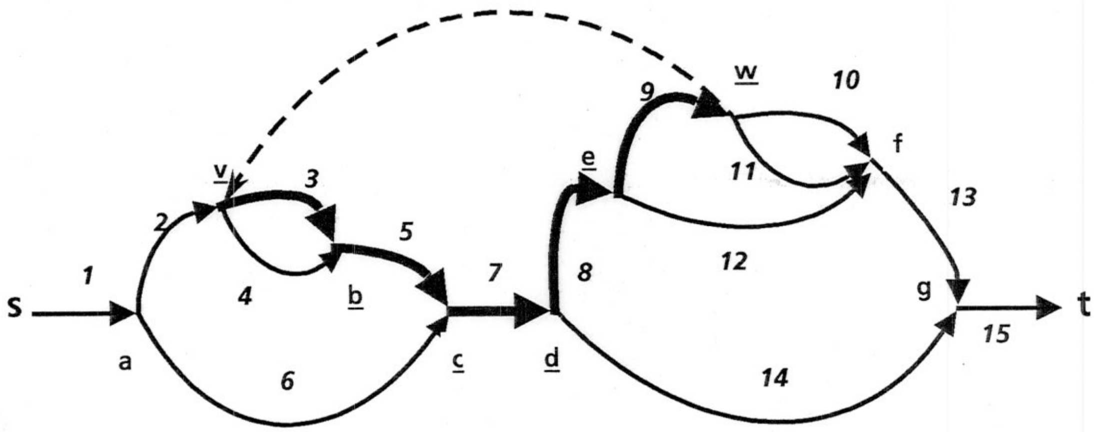


Figure 7a. A BTTSP multidigraph G_B; the backedge is indicated by the dotted line.

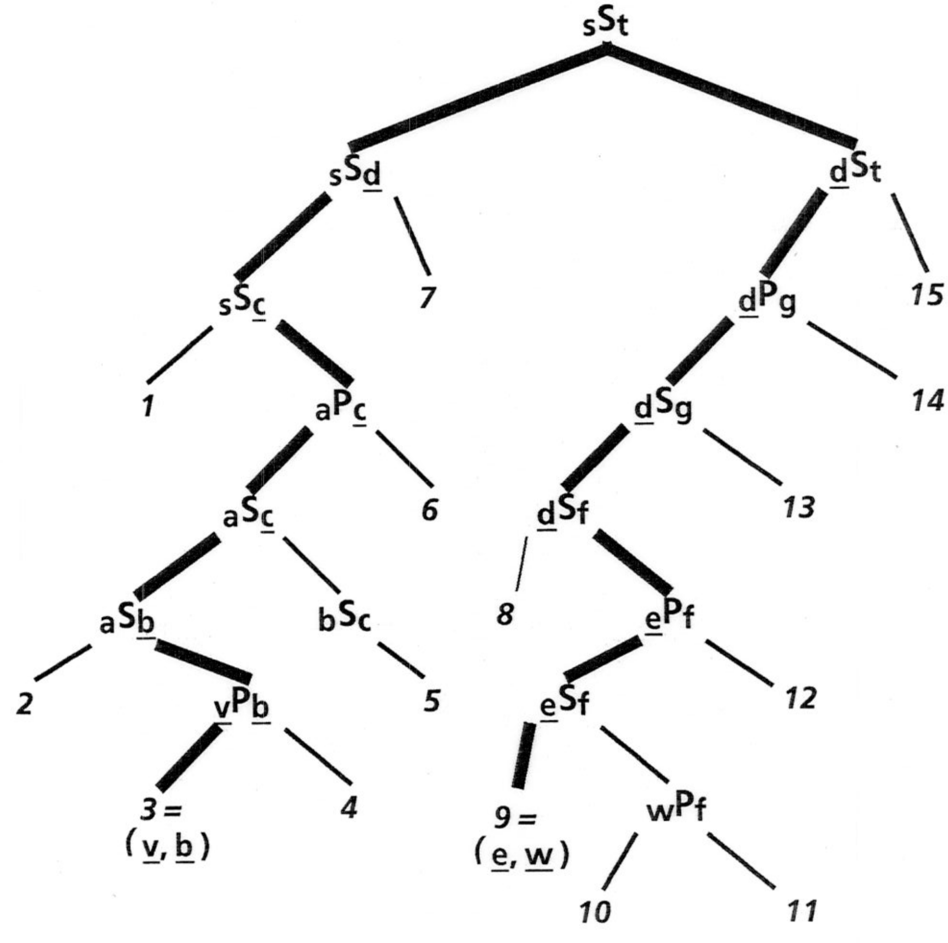


Figure 7b. A binary decomposition tree BDT(G).
The wide solid line corresponds to the path from v to w in Fig. 7a.

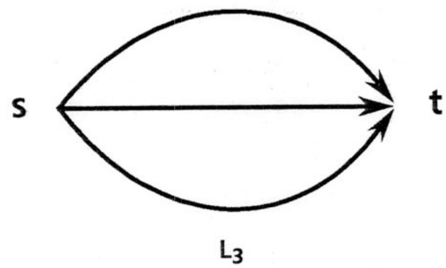


Figure 7c. $|\alpha_{st}(L_3)| \leq 3$.

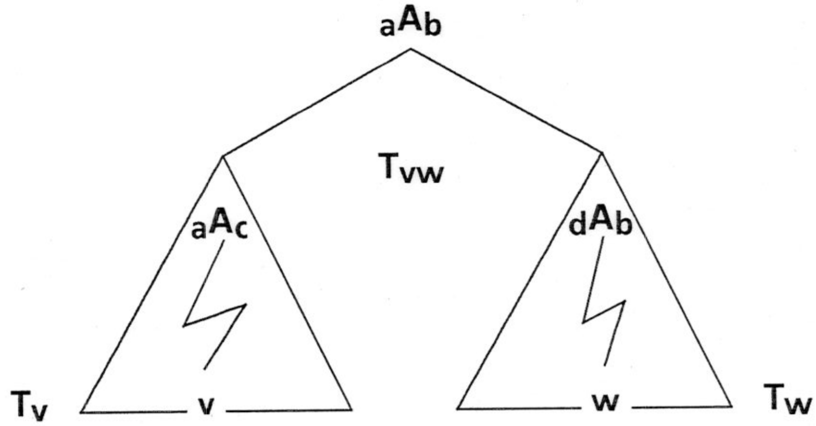


Figure 7d. $_aA_b = {}_aS_b$ and $c = d$. Every path from v to $x \notin T_v$ passes through c. Every path from $x \notin T_w$ to w passes through $d = c$.

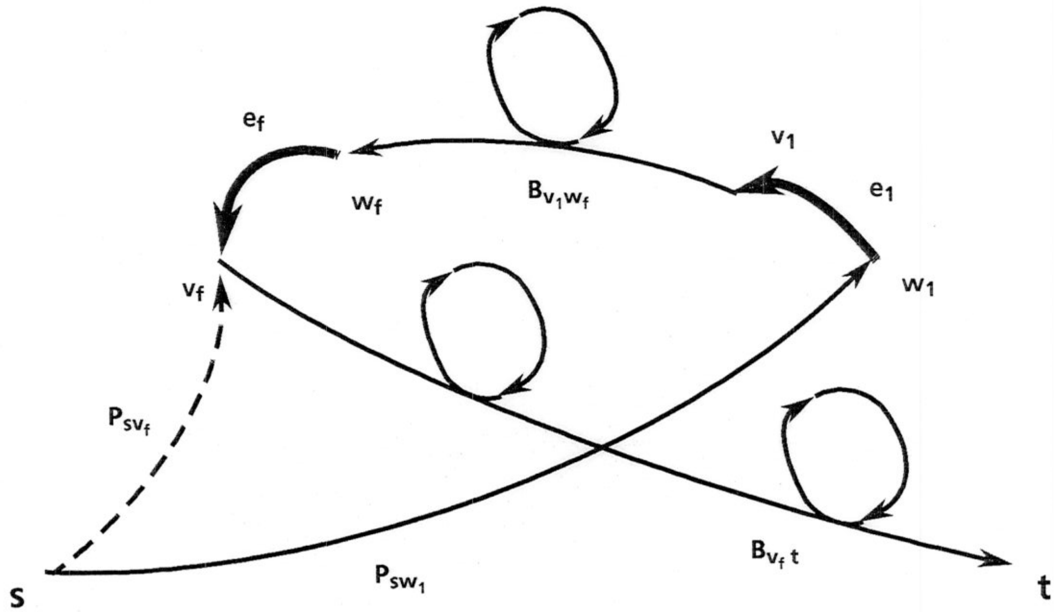


Figure 7e. e_f is the last backedge from which there is a path to w_1. Then we apply the induction hypothesis to the path $P_{sv_f}B_{v_f t}$.