OPINION DYNAMICS IN HETEROGENEOUS NETWORKS: CONVERGENCE CONJECTURES AND THEOREMS*

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Abstract. Recent significant attention has been dedicated to the models of opinion dynamics in which opinions are described by real numbers, and agents update their opinions synchronously by averaging their neighbors' opinions. The neighbors of each agent can be defined as either (1) those agents whose opinions are in its "confidence range," or (2) those agents whose "influence range" contain the agent's opinion. The former definition is employed in Hegselmann and Krause's bounded confidence model, and the latter is novel here. As the confidence and influence ranges are distinct for each agent, the heterogeneous state-dependent interconnection topology leads to a poorly-understood complex dynamic behavior. In both models, we classify the agents via their interconnection topology and, accordingly, compute the equilibria of the system. Then, we define a positive invariant set centered at each equilibrium opinion vector. We show that if a trajectory enters one such set, then it converges to a steady state with constant interconnection topology. This result gives us a novel sufficient condition for both models to establish convergence, and is consistent with our conjecture that all trajectories of the bounded confidence and influence models eventually converge to a steady state under fixed topology.

Key words. opinion dynamics, bounded confidence and influence, social networks, convergence, heterogeneous multi-agent system, leader group

AMS subject classifications. 91D30, 93A14, 37A99, 91C20, 37B55, 91B14

1. Introduction.

Models of social networks are structures made up of individuals that are tied based on their interdependency. Such models explain the confidence or influence flow in populations without relying on detailed social psychological findings. The process of opinion dynamics evolves along the networks of social confidence or influence and affects the structure of the network itself. A common feature among many models of opinion dynamics is bounded confidence or influence, which means that an individual only interacts with those whose opinions are close enough to its own. This idea reflects the psychological concept called *selective exposure* [22]. Broadly defined, "selective exposure refers to behaviors that bring the communication content within reach of one's sensory apparatus" [35]. In the field of social networks, opinion dynamics is of high interest in many areas including: politics, as in voting prediction [1]; physics, as in spinning particles [2]; sociology, as in the diffusion of innovation [31], the electronic exchange of personal information [23], and language change [29, 7]; and finally economics, as in price change [28].

1.1. Literature Review.

The study of opinion dynamics and social networks goes back to the early work by J.R.P. French [9] on "A Formal Theory of Social Power." This work explores the patterns of interpersonal relations and agreements that can explain the influence process in groups of agents. Subsequently, F. Harary provides a necessary and sufficient condition to reach a consensus in French's model of power networks [12]. The modeling of "continuous opinion dynamics", in which opinions are represented by real positive numbers, is initially studied in [27, 6, 15]. In contrast

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to the classical case of "binary opinion dynamics" [32, 10], the continuous case deals with the problem of what happens to the worthiness of a choice or the probability of choosing one decision over another. Recently, *bounded confidence* (BC) models of opinion dynamics, a label coined by Krause in 1998 [16], have received significant attention. BC models are models of continuous opinion dynamics in which agents have bounded confidence in others opinions. The first version of BC models was formulated by Hegselmann and Krause [13], called HK model, where agents synchronously update their opinions by averaging all opinions in their confidence bound. The other popular version of BC models was developed and investigated by Deffuant and Weisbuch [33], called DW model. The HK and DW models are very similar, they differ in their update rule: in DW model a pairwise-sequential updating procedure is employed instead of the synchronized one.

In the HK model, the set of neighbors of the ith agent is defined as those agents whose opinions differ from the ith opinion by less than the ith confidence bound. Hence, this model is dealing with *endogenously* changing topologies, that is, state dependent or changing from inside, in contrast to the *exogenously* changing topologies. For instance, [14, 24, 7, 5] study a synchronized linear averaging model with time-dependent exogenously changing topologies. The HK models are classified based on various factors: a model is called agent- or density-based if its number of agents is finite or infinite, respectively; and a model is called homogeneous or heterogeneous if its confidence bounds are uniform or agent-dependent, respectively. The convergence of both agent- and density-based homogeneous HK models are discussed in [3]. The agent-based homogeneous HK system is proved to reach a fixed state in finite time [8], the time complexity of this convergence is discussed in [21], its stabilization theorem is given in [17], and its rate of convergence to a global consensus is studied in [25]. The heterogeneous HK model is studied by Lorenz who reformulated the HK dynamics as an interactive Markov chain [18] and analyzed the effects of heterogeneous confidence bounds [20]. The convergence of the agent-based heterogeneous HK systems is experimentally observed, but its proof is still an open problem.

1.2. Contributions. In this paper, to distinguish between the HK and DW models, we call a discrete-time agent-based heterogeneous HK model a *synchronized bounded confidence* (*SBC*) model. Additionally, we introduce a model similar to the SBC model and call it the *synchronized bounded influence* (*SBI*) model. The difference is that in an SBI model the set of neighbors of the ith agent is defined as those agents j whose influence range contain the ith agent's opinion. We analyze SBC and SBI models with heterogeneous bounds of confidence or influence, respectively. Indeed, if the SBC and SBI models have agents with homogeneous bounds, then both models are equivalent to the homogeneous HK model. These heterogeneous models of opinion dynamics, in spite of the considerable complexity of their dynamics, describe features of a real society that cannot be explained by homogeneous models. Specifically, the behavior of a heterogeneous opinion dynamics model is richer than that of a homogeneous model in the following ways: (1) an agent may trust an individual but not be trusted back by that same individual; (2) in steady state, an agent can keep its own opinion constant while listening to dissimilar opinions, (whereas, in the steady state of a homogeneous society, any two agents are either disconnected or in consensus); (3) one can observe *pseudo-stable* configurations, i.e., "configurations that have a subset of agents that is stationary, and the rest are the reason for further dynamics" [30]; (4) it is possible for two disconnected agents to reconnect, and an agent with a large bound of confidence or influence can pull clusters of agents towards or away

from each other; (5) the order of opinions is not preserved along the system evolution; (6) "given an average level of confidence, diversity of bounds of confidence enhances the chances for consensus" [20]; and (7) convergence in infinite time is possible.

Based on numerical evidence, we formulate our main conjecture that along the evolution of an SBC or SBI system there exists a finite time, after which the topology of the interconnection network remains unchanged, and as a result, the trajectory converges to a limiting opinion vector. We also observe that each trajectory either reaches a fixed state in finite time or exhibits a pseudo-stable behavior. This observation is verified assuming that the main conjecture is true. Furthermore, the following results put together partly prove our main conjecture: (1) We design an appropriate classification of agents in both SBC and SBI systems. This classification is a function of state-dependent interconnection topology of the system, and can explain the observed pseudo-stable behavior. (2) We introduce the new notion of *final value at constant topology*, and based on our classification, we formulate the map under which this value is an image of the current opinion vector. The set of final values at constant topology is a superset of the equilibria of the system. We derive necessary and sufficient conditions for the final value at constant topology to be an equilibrium vector. (3) For each equilibrium opinion vector, we define its *equi-topology neighborhood* and *invariant equi-topology neighborhood*. We show that if a trajectory enters the invariant equi-topology neighborhood of an equilibrium vector, then it remains confined to its equi-topology neighborhood, and sustains an interconnection topology equal to that of the equilibrium vector. This fact establishes a novel and simple sufficient condition under which: the initial opinion vector converges to a steady state; the topology of the interconnection network remains unchanged; and the limiting opinion vector is equal to the final value at constant topology of the initial opinion vector. (4) We explore some interesting behavior of classes of agents when they update their opinions under fixed interconnection topology for infinite time. For instance, we compute agents rates and directions of convergence, and show the existence of a leader group for each group of agents that determines the follower's rate and direction of convergence.

In our extensive simulation results, we observe that for uniformly randomly generated initial opinion vector and bounds vector, the SBC and SBI trajectories eventually satisfy our novel sufficient condition for convergence with probability one. We give some intuitive explanation for this observation. Finally, we conjecture that the SBI trajectories reach a fixed state in finite time more often than the SBC trajectories. To substantiate this conjecture, we present a sufficient condition for SBC and SBI systems separately that guarantees reaching an *agreement opinion vector*, which occurs in finite time and is a general case of reaching a global consensus. Based on these sufficient conditions, we explain our last conjecture.

1.3. Organization. This paper is organized as follows. In Section 2, the mathematical models, conjectures, agents classification, and spectral properties of the adjacency matrices are presented. In Section 3, the final value at constant topology is introduced and characterized. Section 4 contains the novel sufficient condition for constant topology and convergence, and for convergence in finite time. In Section 5, the simulation results and intuitive explanations are presented. In Section 6, the behavior of the system assuming that its interconnection topology remains unchanged in a long run is analyzed. Finally, Section 7 contains the conclusion and open questions.

2. Mathematical Models. Consider n interacting agents and assume that each agent's opinion is expressed by a real number, say y_i for agent $i \in \{1, \dots, n\}$. In *bounded confidence interaction*, the opinion y_i is affected by the opinion y_j if $|y_i - y_j| \leq$

r_i, where the positive number r_i is the *confidence bound* of agent i. In *bounded influence interaction*, the opinion y_i is affected by the opinion y_j if $|y_i - y_j| \leq r_j$, where the positive number r_j is the *influence bound* of agent j. The *opinion vector* $y \in \mathbb{R}^n$ and the *bounds vector* $r \in \mathbb{R}^n_{>0}$ are obtained by stacking all y_i's and r_i's, respectively. We associate to each opinion vector y two digraphs, both with nodes $\{1, \ldots, n\}$ and edge set defined as follows: denoting the set of out-neighbors of node i by $\mathcal{N}_i(y)$

- in a *synchronized bounded confidence* (SBC) digraph, $\mathcal{N}_i(y) = \{j \in \{1, \ldots, n\} : |y_i - y_j| \leq r_i\}$; and
- in a *synchronized bounded influence* (SBI) digraph, $\mathcal{N}_i(y) = \{j \in \{1, \ldots, n\} : |y_i - y_j| \leq r_j\}$.

We let $G_r(y)$ denote one of the two *proximity digraphs*, its precise meaning being clear from the context.

We associate to the SBC and SBI digraphs two dynamical systems, called the *SBC* and *SBI systems* respectively. Both dynamical systems update a trajectory $x : \mathbb{N} \to \mathbb{R}^n$ according to the discrete-time and continuous-state rule

$$x(t+1) = A(x(t))x(t), \tag{2.1}$$

where the i, j entry of the *adjacency matrix* $A(y) \in \mathbb{R}^{n \times n}$ for any $y \in \mathbb{R}^n$ is defined by

$$a_{ij}(y) = \begin{cases} \frac{1}{|\mathcal{N}_i(y)|}, & \text{if } j \in \mathcal{N}_i(y), \\ 0, & \text{if } j \notin \mathcal{N}_i(y), \end{cases}$$

and $|\mathcal{N}_i(y)|$ is the cardinality of $\mathcal{N}_i(y)$. Note that $i \in \mathcal{N}_i(y)$, in other words, every agent has some self-confidence or self-influence. This assumption is a key factor in the convergence of infinite products of adjacency matrices [19]. In the following, we present our conjectures on SBC and SBI systems, and the trajectories of Figure 2.1 support these conjectures.

Conjecture 2.1 (Existence of a limiting opinion vector). *Every trajectory of an SBC or SBI system converges to a limiting opinion vector.*

Conjecture 2.2 (Constant-topology in finite time). *For any trajectory $x(t)$ of an SBC or SBI system, there exists a finite time τ after which the state-dependent interconnection topology, or equivalently $G_r(x(t))$, remains constant.*

Before proceeding, let us define a term borrowed from [30]. A trajectory $x(t) \in \mathbb{R}^n$ that is converging to limiting opinion vector $x_\infty \in \mathbb{R}^n$ is said to have a *pseudo-stable* behavior after τ, if the node set $\mathcal{V} = \{1, \ldots, n\}$ is composed of two non-empty subsets $\mathcal{V}_{\text{fixed}}$ and $\mathcal{V}_{\text{converging}}$ such that, for all $t \geq \tau$,

$$\begin{cases} x_i(t) = x_{\infty,i}, & \text{if } \ i \in \mathcal{V}_{\text{fixed}}, \\ x_i(t) < x_i(t+1) < x_{\infty,i} \ \text{ or } \ x_i(t) > x_i(t+1) > x_{\infty,i}, & \text{if } \ i \in \mathcal{V}_{\text{converging}}. \end{cases} \tag{2.2}$$

Conjecture 2.3 (Pseudo-stable behavior). *For any SBC or SBI trajectory, there exists a finite time after which the trajectory either reaches a fixed state or exhibits a pseudo-stable behavior.*

Conjecture 2.4 (Convergence of SBI systems versus SBC systems). *For any initial opinion vector and bounds vector that are generated uniformly randomly, the SBI system is more likely to converge in finite time than the SBC system.*

This paper aims to study these conjectures.

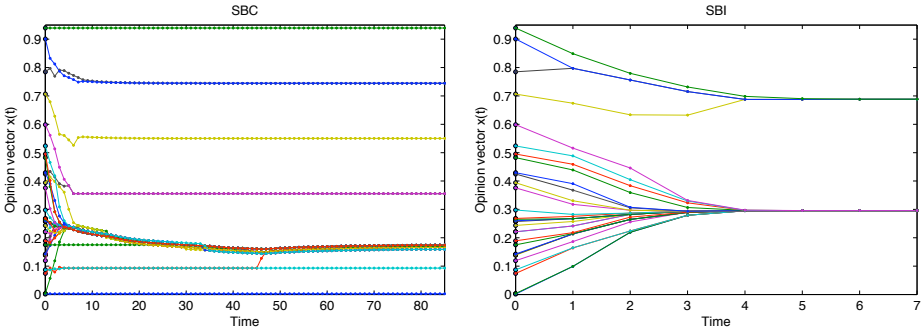

FIG. 2.1. *The trajectory of an SBC system (left) and an SBI system (right) are illustrated. Both systems have the same initial opinion vector and bounds vectors that are randomly generated. However, the SBI trajectory reaches a fixed state in six time steps, while the SBC trajectory converges in infinite time. The interconnection topology of the agents in the SBC system remains constant after $t = 64$, and hence its trajectory exhibits a pseudo-stable behavior.*

2.1. Agents Classification. In this section, we introduce a classification of agents for both SBC and SBI systems based on their state-dependent interaction topology at each time step. This classification is used later to find the limiting opinion vector and explain the pseudo-stable behavior. First, let us quote some relevant definitions from graph theory, e.g. see [4]. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A digraph is *strongly connected* if every node is globally reachable. A digraph is *weakly connected* if replacing all of its directed edges with undirected edges produces a connected undirected graph. A maximal subgraph which is strongly or weakly connected forms a *strongly connected component* (SCC) or a *weakly connected component* (WCC), respectively. Every digraph G can be decomposed into either its SCC's or WCC's. Accordingly, the *condensation digraph* of G, denoted $C(G)$, is defined as follows: the nodes of $C(G)$ are the SCC's of G, and there exists a directed edge in $C(G)$ from node H_1 to node H_2 if and only if there exists a directed edge in G from a node of H_1 to a node of H_2. A node with out-degree zero is named a *sink*. Knowing that the condensation digraphs are acyclic, each WCC of $C(G)$ is also acyclic and thus has at least one sink. In a digraph, i is a *predecessor* of j and j is a *successor* of i if there exists a directed path from node i to node j.

For opinion vector $y \in \mathbb{R}^n$, let $G_r(y)$ denote either of its SBC or SBI digraphs. We classify the SCC's of $G_r(y)$ into three classes. An SCC of $G_r(y)$ is called a *closed-minded component* if it is a complete subgraph of $G_r(y)$ and corresponds to a sink of $C(G_r(y))$. An SCC of $G_r(y)$ is called a *moderate-minded component* if it is a non-complete subgraph of $G_r(y)$ and corresponds to a sink of $C(G_r(y))$. The rest of SCC's of $G_r(y)$ are called *open-minded SCC's*. Now, the *open-minded subgraph* of $G_r(y)$ is the remaining subgraph after removing $G_r(y)$'s closed- and moderate-minded components and their edges. A WCC of the open-minded subgraph of $G_r(y)$ will be called an *open-minded WCC*, see Figure 2.2.

REMARK 2.5. *Previously, (Lorenz, 2006) classified the agents of an SBC system into two classes of* essential *and* inessential. *An agent is essential if any of its successors is also a predecessor, and an agent is inessential if it has a successor who is not a predecessor [19]. This classification is similar to the one used for Markov chains [26, Chapter 1.2]. It is easy to see that the closed- and moderate-minded components*

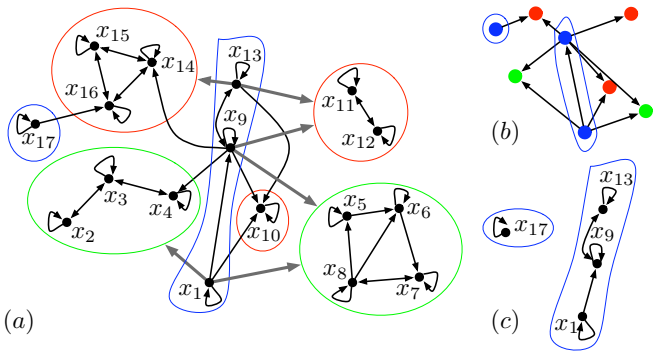

FIG. 2.2. *Consider the opinion vector* $x = [0.1\ 0.24\ 0.27\ 0.3\ 0.34\ 0.37\ 0.39\ 0.4\ 0.5\ 0.6\ 0.67\ 0.68\ 0.75\ 0.85\ 0.86\ 0.87\ 1]^T$ *and bounds vector* $r = [0.5\ 0.04\ 0.04\ 0.04\ 0.031\ 0.021\ 0.011\ 0.061\ 0.25\ 0.01\ 0.04\ 0.03\ 0.3\ 0.07\ 0.07\ 0.07\ 0.135]^T$*: (a) shows the SBC digraph of* x*,* $G_r(x)$*, with its closed- (red), moderate- (green), and open-minded (blue) components, and each thick gray edge represents multiple edges to all agents in one component; (b) shows the condensation digraph of* $G_r(x)$*; and (c) shows the open-minded subgraph of* $G_r(x)$ *that is composed of two open-minded WCC's.*

are in essential class, and the open-minded components are inessential.

2.2. Spectral Properties of Adjacency Matrix. For any opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system (2.1), the adjacency matrix $A(y)$ is a non-negative row-stochastic matrix, and its nonzero diagonal establishes its aperiodicity. Since $C(G_r(y))$ is an acyclic digraph, its adjacency matrix is lower-triangular in an appropriate ordering [4]. In such ordering, the adjacency matrix of $G_r(y)$ is lower block triangular. Based on the classification of the SCC's in $G_r(y)$, we put $A(y)$ into the *canonical form* $\overline{A}(y)$, by an appropriate *canonical permutation matrix* $P(y)$,

$$\overline{A}(y) = P(y)A(y)P^T(y) = \begin{bmatrix} C(y) & 0 & 0 \\ 0 & M(y) & 0 \\ \Theta_C(y) & \Theta_M(y) & \Theta(y) \end{bmatrix}. \tag{2.3}$$

The submatrices $C(y)$, $M(y)$, and $\Theta(y)$ are block diagonal. Each diagonal block $C_i(y)$, with size $n_i(y)$, is the adjacency matrix of the ith closed-minded component, and is equal to $C_i(y) = \mathbf{1}_{n_i(y)}\mathbf{1}_{n_i(y)}^T/n_i(y)$. Let us call a matrix with such structure a *complete consensus matrix*, whose spectrum is found to be $\{1, 0, \ldots, 0\}$. Similarly, each diagonal block $M_i(y)$ is the adjacency matrix of the ith moderate-minded component. Each entry in $\Theta_C(y)$ or $\Theta_M(y)$ represents an edge from an open-minded node to a closed- or moderate-minded node, respectively. Finally, in the submatrix $\Theta(y)$, each diagonal block $\Theta_i(y)$ corresponds to one open-minded WCC, and is block lower triangular and strictly row-substochastic. By strictly row-substochastic we mean a square matrix with nonnegative entries so that every row adds up to at most one, and there exists at least one row whose sum is strictly less than one. Note that the adjacency matrix of each SCC in $G_r(y)$ is a diagonal block of $\overline{A}(y)$ and is row-stochastic, nonnegative and primitive. On account of the properties of the open-minded class, the following lemma is proved.

LEMMA 2.6. *For any row* k *of the submatrix* $\Theta(y)$*, there exists* $p_k \in \mathbb{N}$ *such that the* k*th row sum of* $\Theta(y)^{p_k}$ *is strictly less than 1.*

Proof. Every WCC of $C(G_r(y))$ contains at least one sink. Hence, from any open-minded agent k, there exists a directed path of length p_k to an agent s in either

a closed- or moderate-minded component. Now, consider the canonical adjacency matrix to the power p_k,

$$\overline{A}(y)^{p_k} = \begin{bmatrix} C(y)^{p_k} & 0 & 0 \\ 0 & M(y)^{p_k} & 0 \\ \Theta_C^{(p_k)}(y) & \Theta_M^{(p_k)}(y) & \Theta(y)^{p_k} \end{bmatrix}.$$

Existence of such directed path, by [4, Lemma 1.32], implies that the (k, s) entry of $\overline{A}(y)^{p_k}$, which belongs to either of the submatrices $\Theta_C^{(p_k)}(y)$ or $\Theta_M^{(p_k)}(y)$, is nonzero. Consequently, the kth row sum of $\Theta(y)^{p_k}$ is strictly less than 1. □

It follows from Lemma 2.6 that $\lim_{t\to\infty} \Theta(y)^t = \mathbf{0}$, for a proof of which refer to [26, Theorem 4.3]. Therefore, the spectral radius of $\Theta(y)$ is strictly less than one.

EXAMPLE 2.7. *Consider the SBC system of Figure 2.2 with the permuted opinion vector* $x = [x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_{10}\ x_{11}\ x_{12}\ x_{14}\ x_{15}\ x_{16}\ x_{17}\ x_9\ x_1]^T$. *Then, the canonical form of the adjacency matrix* $A(x)$ *contains the following submatrices:*

$$C(x) = \begin{bmatrix} 1 & & & & & & & & & & & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & & & & & & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & & & & & & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & & & & & & & \end{bmatrix}, \quad M(x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & & & & & & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & & & \\ & & & \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & \\ & & & & & \frac{1}{2} & \frac{1}{2} & & & & & & & & & \\ & & & & & \frac{1}{2} & \frac{1}{2} & & & & & & & & & \\ & & & & & & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & & \\ & & & & & & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & & \end{bmatrix},$$

$$\Theta(x) = \begin{bmatrix} \frac{1}{2} & & & & & & & & & & & & & & & \\ & \frac{1}{8} & \frac{1}{8} & & & & & & & & & & & & & \\ & \frac{1}{11} & \frac{1}{11} & & & & & & & & & & & & & \\ & & \frac{1}{11} & & & & & & & & & & & & & \\ & & \frac{1}{10} & \frac{1}{10} & & & & & & & & & & & & \end{bmatrix}, \quad \Theta_C(x) = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & & & & & & & & & & \\ \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & & \frac{1}{8} & & & & & & & & & & \\ \frac{1}{10} & & & & & & & & & & & & & & & \end{bmatrix},$$

$$\Theta_M(x) = \begin{bmatrix} 0 & & & \dots & & & 0 & & & & & & & & & \\ 0 & & & \dots & & & 0 & & & & & & & & & \\ & & & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & & & & & & & & & \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & & & & & & & & & \end{bmatrix}.$$

3. Equilibria and Final Value at Constant Topology. An opinion vector y_0 is an *equilibrium opinion vector* of the dynamical system (2.1) if and only if y_0 is an eigenvector of the adjacency matrix $A(y_0)$ for the eigenvalue one or, equivalently, $y_0 = A(y_0)y_0$. Next, based on Conjecture 2.2, we introduce the following definition.

DEFINITION 3.1 (Final value at constant topology). *For any opinion vector* $y \in \mathbb{R}^n$ *we define its* final value at constant topology fvct : $\mathbb{R}^n \to \mathbb{R}^n$ *to be the limiting opinion vector of an SBC or SBI system whose initial opinion vector is* y *and the interconnection topology of its agents remains unchanged for all* $t \geq 0$. *That is,*

$$\text{fvct}(y) = \lim_{t\to\infty} A(y)^t y \in \mathbb{R}^n.$$

The final value at constant topology of any equilibrium opinion vector is equal to itself, that is, $\text{fvct}(y_0) = \lim_{t\to\infty} A(y_0)^t y_0 = y_0$. Therefore, the set of final values at

constant topology is a superset of the equilibria of the system, and the set of equilibria is a superset of the limiting opinion vectors. The condition under which a final value at constant topology is an equilibrium is discussed as follows.

PROPOSITION 3.2 (Properties of the final value at constant topology). *For any opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system, whose adjacency matrix can be found from equation (2.3):*

(i) $\mathrm{fvct}(y)$ *is well defined, and is equal to*

(3.1)
$$\mathrm{fvct}(y) = P^T(y) \begin{bmatrix} C & 0 & 0 \\ 0 & M^* & 0 \\ (I-\Theta)^{-1}\Theta_C C & (I-\Theta)^{-1}\Theta_M M^* & 0 \end{bmatrix} (y)P(y)y,$$

where the submatrix $M^*(y)$ is set equal to $\lim_{t\to\infty} M(y)^t$ and is well defined.

(ii) If the two networks of agents with opinion vectors y and $\mathrm{fvct}(y)$ have the same interconnection topology or, equivalently, $G_r(y) = G_r(\mathrm{fvct}(y))$, then

(a) $\mathrm{fvct}(y)$ *is an equilibrium opinion vector,*

(b) $G_r(y)$ *contains no moderate-minded component, and*

(c) in any WCC of $G_r(\mathrm{fvct})$, the maximum and minimum opinions belong to its closed-minded components.

Proof. Let us first drop the y argument for matrices for readability. Regarding part (i),

$$\mathrm{fvct}(y) = \lim_{t\to\infty} A^t y = P^T \lim_{t\to\infty} \overline{A}^t P y = P^T \lim_{t\to\infty} \begin{bmatrix} C^t & 0 & 0 \\ 0 & M^t & 0 \\ \Theta_C^{(t)} & \Theta_M^{(t)} & \Theta^t \end{bmatrix} Py.$$

From Section 2.2, $C^t = C$ for any $t \geq 1$, $\lim_{t\to\infty} \Theta^t = \mathbf{0}$, and each diagonal block M_i, with size n_i, is a row-stochastic primitive nonnegative matrix. For such matrices the Perron-Frobenius Theorem tells us that the spectral radius is equal to one, and the essential spectral radius is strictly less than one. Thus, if we let $\nu_i \in \mathbb{R}^{n_i}$ be a left eigenvector of M_i for the eigenvalue one, then from [4, Remark 1.69],

$$M_i^* = \lim_{t\to\infty} M_i^t = (\nu_i \mathbf{1}_{n_i})^{-1} \mathbf{1}_{n_i} \nu_i. \tag{3.2}$$

Using the solution to the infinite products of transition matrices of a Markov chain, given in [11, Chapter 5], it can be shown that $\lim_{t\to\infty} \Theta_C^{(t)} = (I-\Theta)^{-1}\Theta_C C$, and $\lim_{t\to\infty} \Theta_M^{(t)} = (I-\Theta)^{-1}\Theta_M M^*$.

Regarding part (ii)a, $G_r(y) = G_r(\mathrm{fvct}(y))$ results in $A(y) = A(\mathrm{fvct}(y))$, and hence

$$A(\mathrm{fvct}(y))\,\mathrm{fvct}(y) = A(y) \lim_{t\to\infty} A(y)^t y = \lim_{t\to\infty} A(y)^t y = \mathrm{fvct}(y).$$

Regarding part (ii)b, by contradiction assume that $G_r(y)$ contains at least one moderate-minded component with the opinion vector y_{M_1} and the adjacency matrix M_1. The trajectory of each sink in $C(G_r(y))$ is independent of other nodes, hence $\mathrm{fvct}_{M_1}(y) = \lim_{t\to\infty} M_1^t y_{M_1}$, and by equation (3.2), $\mathrm{fvct}_{M_1}(y) = (\nu_1 \mathbf{1}_{n_1})^{-1} \mathbf{1}_{n_1} \nu_1 y_{M_1}$. Since $(\nu_1 \mathbf{1}_{n_1})^{-1} \nu_1 y_{M_1}$ is a scalar, all agents in one moderate-minded component are in consensus in final value at constant topology, and their adjacency matrix is no longer M_1, but rather a complete consensus matrix, which contradicts the assumption of $G_r(y) = G_r(\mathrm{fvct}(y))$.

Regarding part (ii)c, let i denote the agent with the minimum final value at constant topology in one WCC of $G_r(\mathrm{fvct}(y))$. By contradiction, assume that i is

open-minded. Granted that the confidence or influence bounds are strictly greater than zero, in the set of out-neighbors of each open-minded agent there exists at least one agent with distinct opinion. Since i has the smallest opinion among its neighbors, $\text{fvct}_i(y)$ increases after taking an average of i's out-neighbors opinions. In other words, the ith entry in the vector $A(\text{fvct}(y))\,\text{fvct}(y)$ is strictly larger than $\text{fvct}_i(y)$, which contradicts the fact that $\text{fvct}(y)$ is an equilibrium opinion vector and invariant under matrix $A(\text{fvct}(y))$. Same can be proved for the agent with the maximum opinion. □

4. Convergence Analysis. In this section, motivated by Conjectures 2.1 and 2.2, we drive sufficient condition which guarantees that an SBC or SBI trajectory converges to a limiting opinion vector. Next, to explain Conjecture 2.4, we study sufficient conditions for SBC and SBI systems separately that guarantee reaching a fixed state in finite time.

4.1. Sufficient Condition for Constant Topology and Convergence. Our sufficient condition is based on specific neighborhoods of each opinion vector, which is introduced in the following.

DEFINITION 4.1 (Equi-topology distances and neighborhoods). *Consider an SBC or SBI system with opinion vector $z \in \mathbb{R}^n$.*

(i) The equi-topology distance *of z is a vector of non-negative entries $\epsilon(z) \in \mathbb{R}^n_{\geq 0}$ defined by, for $i \in \{1, \dots, n\}$,*

$$\epsilon_i(z) = 0.5 \min\{||z_i - z_j| - R| : j \in \{1, \dots, n\} \setminus \{i\}, R \in \{r_i, r_j\}\}, \tag{4.1}$$

and the equi-topology neighborhood *of z is the set $\mathcal{B}_{\text{et}}(z)$ of opinion vectors $y \in \mathbb{R}^n$ such that*

$$\begin{aligned} &|y_i - z_i| < \epsilon_i(z), \text{ for all } i \in \{1, \dots, n\} \text{ with } \epsilon_i(z) > 0, \text{ and} \\ &|y_i - z_i| = \epsilon_i(z), \text{ for all } i \in \{1, \dots, n\} \text{ with } \epsilon_i(z) = 0. \end{aligned}$$

(ii) The invariant equi-topology distance *of z is the vector of non-negative entries $\delta(z) \in \mathbb{R}^n_{\geq 0}$ defined by, for $i \in \{1, \dots, n\}$,*

$$\delta_i(z) = \min\{\epsilon_j(z) : j \text{ is a predecessor of } i \text{ in the graph } G_r(z)\}, \tag{4.2}$$

and the invariant equi-topology neighborhood *of z is the set $\mathcal{B}_{\text{iet}}(z)$ of opinion vectors $y \in \mathbb{R}^n$ such that*

$$\begin{aligned} &|y_i - z_i| < \delta_i(z), \text{ for all } i \in \{1, \dots, n\} \text{ with } \delta_i(z) > 0, \text{ and} \\ &|y_i - z_i| = \delta_i(z), \text{ for all } i \in \{1, \dots, n\} \text{ with } \delta_i(z) = 0. \end{aligned}$$

Note that in any SBC or SBI digraph, each node has a self-loop, and hence each agent is a predecessor of itself. Therefore, for any opinion vector $z \in \mathbb{R}^n$ and for all $i \in \{1, \dots, n\}$, we have $\delta_i(z) \leq \epsilon_i(z)$, which results in $\mathcal{B}_{\text{iet}}(z) \subset \mathcal{B}_{\text{et}}(z)$.

LEMMA 4.2 (Sufficient condition for equal topologies). *Consider an SBC or SBI system with opinion vectors $y, z \in \mathbb{R}^n$. If y belongs to the equi-topology neighborhood of z, then the two networks of agents with opinion vectors y and z have the same interconnection topology, or equivalently $G_r(y) = G_r(z)$.*

REMARK 4.3. *For any $y \in \mathbb{R}^n$, if $y \in \mathcal{B}_{\text{et}}(\text{fvct}(y))$, then by Lemma 4.2 we have $G_r(y) = G_r(\text{fvct}(y))$. Hence, by Proposition 3.2, $\text{fvct}(y)$ is an equilibrium opinion vector, and $G_r(y)$ contains no moderate-minded component.*

Proof. For any $i, j \in \{1, \ldots, n\}$ two cases exists:
1. j is an out-neighbor of i in $G_r(z)$, hence $|z_i - z_j| \leq r$, where in an SBC system $r = r_i$ and in an SBI system $r = r_j$. In either system, since $y \in \mathcal{B}_{\text{et}}(z)$,

$$|y_i - y_j| \leq |z_i - z_j| + \epsilon_i(z) + \epsilon_j(z) \leq |z_i - z_j| + ||z_i - z_j|| - r| = r.$$

2. j is not an out-neighbor of i in $G_r(z)$, hence $|z_i - z_j| > r$, with r defined above. If both $\epsilon_i(z)$ and $\epsilon_j(z)$ are zero, then $y \in \mathcal{B}_{\text{et}}(z)$ gives us

$$|y_i - y_j| = |z_i - z_j| > r,$$

and if at least one is nonzero, then

$$|y_i - y_j| > |z_i - z_j| - \epsilon_i(z) - \epsilon_j(z) \geq |z_i - z_j| - ||z_i - z_j| - r| = r.$$

Therefore, the neighboring relation of agents in $G_r(z)$ is preserved in $G_r(y)$. One can also prove that any neighboring relation in $G_r(y)$ is preserved in $G_r(z)$. ∎

THEOREM 4.4 (Sufficient condition for constant topology and convergence). *Consider a trajectory $x(t)$ of an SBC or SBI system. Assume that there exists an equilibrium opinion vector $z \in \mathbb{R}^n$ for the system such that $x(0) \in \mathbb{R}^n$ belongs to the invariant equi-topology neighborhood of z. Then, for all $t \geq 0$:*

(i) $x(t)$ takes value in the equi-topology neighborhood of z, and hence $G_r(z) = G_r(x(t))$;
(ii) $G_r(x(t))$ contains no moderate-minded component; and
(iii) $x(t)$ converges to $\text{fvct}(x(0))$ as time goes to infinity.

REMARK 4.5 (Interpretation of Theorem 4.4). *This theorem tells us that if the trajectory of an SBC or SBI system enters a specific ball around any equilibrium opinion vector of that system, then it remains in some larger ball around that vector for all future iterations. Moreover, the proximity digraph of the trajectory and the equilibrium opinion vector remain equal.*

REMARK 4.6. *Under the condition of Theorem 4.4, a trajectory $x(t)$ converges to its final value at constant topology $\text{fvct}(x(t)) = \text{fvct}(x(0))$. However, $\text{fvct}(x(t))$ is not necessarily equal to the equilibrium opinion vector z, and the proximity digraphs $G_r(\text{fvct}(x(t)))$ and $G_r(x(t))$ can be different, see Figure 4.1.*

REMARK 4.7. *One special case of Theorem 4.4 is when $x(0) \in \mathcal{B}_{\text{iet}}(\text{fvct}(x(0)))$, which implies that $\text{fvct}(x(0))$ is an equilibrium opinion vector, again see Figure 4.1.*

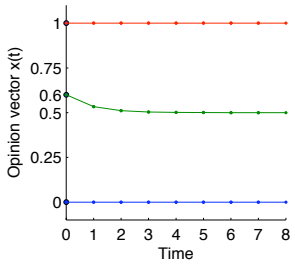

FIG. 4.1. *The SBC trajectory $x(t)$ with $x(0) = [0 \quad 0.6 \quad 1]^T$ and confidence bounds $r = [0.25 \quad 1 \quad 0.25]^T$ is illustrated. It can be computed that $\text{fvct}(x(0)) = [0 \quad 0.5 \quad 1]$ and $\delta(\text{fvct}(x(0))) = [0.25 \quad 0.25 \quad 0.25]$. Clearly, $x(0) \in \mathcal{B}_{\text{iet}}(\text{fvct}(x(0)))$, i.e., the initial vector satisfies the special case of Theorem 4.4 stated in Remark 4.7. Hence, $x(t)$ converges to $\text{fvct}(x(0))$, and their proximity digraphs are equal. However, if the confidence bounds are equal to $r = [0.5 \quad 1 \quad 0.25]^T$, then $\delta(\text{fvct}(x(0))) = [0 \quad 0 \quad 0]$, and $x(t) \notin \mathcal{B}_{\text{iet}}(\text{fvct}(x(0)))$ for all $t \geq 0$. Therefore, $x(t)$ converges to $\text{fvct}(x(0))$, while their proximity digraphs are different. Both trajectories with the two confidence bounds vectors are the same.*

Proof. [Proof of Theorem 4.4] Regarding statement (i), by induction we prove that $x(t) \in \mathcal{B}_{\text{et}}(z)$ for all $t \geq 0$, which by Lemma 4.2 results in $G_r(x(t)) = G_r(z)$. The first induction step is $x(0) \in \mathcal{B}_{\text{et}}(z)$, which is true knowing that $x(0) \in \mathcal{B}_{\text{iet}}(z)$ and

$\mathcal{B}_{\text{iet}}(z) \subset \mathcal{B}_{\text{et}}(z)$. To complete the induction argument, assume that the statement (i) holds at times $t = 0, \ldots, \tau$, which implies that $A(z) = A(x(t))$. The equilibrium opinion vector z satisfies $z = A(z)z$, thus we have $x(t+1) - z = A(x(t))x(t) - A(z)z = A(z)(x(t) - z)$ or, equivalently

$$x_i(t+1) - z_i = \frac{1}{|\mathcal{N}_i(z)|} \sum_{j \in \mathcal{N}_i(z)} (x_j(t) - z_j). \tag{4.3}$$

One can see that

$$\begin{aligned} |x_i(\tau+1) - z_i| \leq \max_{j \in \mathcal{N}_i(z)} |x_j(\tau) - z_j| \leq \max_{\ell \in \mathcal{N}_j(z), j \in \mathcal{N}_i(z)} |x_\ell(\tau-1) - z_\ell| \\ \leq \cdots \leq \max_{k \in \mathcal{M}} |x_k(0) - z_k|, \end{aligned} \tag{4.4}$$

where $\mathcal{M}$ is a subset of successors of i in $G_r(z)$, and thus for any $k \in \mathcal{M}$, equation (4.2) tells us that $\delta_k(z) \leq \epsilon_i(z)$. Here again two cases exists: First, if for all $k \in \mathcal{M}$, $\delta_k(z) = 0$, then the condition $x(0) \in \mathcal{B}_{\text{iet}}(z)$ implies that $x_k(0) - z_k = 0$, and it follows from inequality (4.4) that $x_i(\tau+1) - z_i = 0$. Second, if there exists $\ell \in \mathcal{M}$ such that $\delta_\ell(z) > 0$, then $\epsilon_i(z) > 0$ and

$$|x_i(\tau+1) - z_i| \leq \max_{k \in \mathcal{M}} |x_k(0) - z_k| < \max_{k \in \mathcal{M}} \delta_k(z) \leq \epsilon_i(z).$$

Therefore, $x(\tau+1) \in \mathcal{B}_{\text{et}}(z)$.

Regarding statement (ii), according to Section 3, an equilibrium opinion vector is equal to its own final value at constant topology. Hence $G_r(z) = G_r(z^*(z))$, and by Proposition 3.2, $G_r(z)$ and thus $G_r(x(t))$ contain no moderate-minded component.

Regarding statement (iii), according to the definition of the final value at constant topology, if the topology remains constant for all $t \geq 0$, then $x(t)$ converges to $\text{fvct}(x(0))$. ∎

Motivated by Conjecture 2.1, the existence of a limiting opinion vector is required in the following lemma.

LEMMA 4.8 (Sufficient condition for a limiting opinion vector to be an equilibrium). *Pick a trajectory $x(t)$ of an SBC or SBI system that is convergent. Denote the limiting opinion vector of $x(t)$ by x_∞. If $\min_{i \in \{1,\ldots,n\}} \epsilon_i(x_\infty) > 0$, where $\epsilon(x_\infty)$ is the equi-topology distance of x_∞ , then there exists time T such that for all $t \geq T$:*

(i) $G_r(x_\infty) = G_r(x(t))$, and

(ii) $x_\infty = \text{fvct}(x(t))$, and is an equilibrium opinion vector.

Proof. According to the definition of convergence, for any $\delta \in \mathbb{R}_{>0}$, there exists T such that for all $t \geq T$, $\|x(t) - x_\infty\|_\infty < \delta$. Now, if we let δ be equal to $\min_{i \in \{1,\ldots,n\}} \epsilon_i(x_\infty)$, then $\|x(t) - x_\infty\|_\infty < \min_{i \in \{1,\ldots,n\}} \epsilon_i(x_\infty)$ for all $t \geq T$, and it follows from Lemma 4.2 that $G_r(x_\infty) = G_r(x(t))$. Under fixed topology, $x(t)$ converges to its final value at constant topology, thus $x_\infty = \text{fvct}(x(t))$. Moreover, the equality $G_r(x(t)) = G_r(\text{fvct}(x(t)))$ tells us that $\text{fvct}(x(t))$, and hence x_∞, is an equilibrium opinion vector. ∎

COROLLARY 4.9. *An equilibrium opinion vector z is a Lyapunov stable equilibrium vector for the system if $\min_{i \in \{1,\ldots,n\}} \epsilon_i(z) > 0$.*

4.2. Sufficient Condition for Convergence in Finite Time and Consensus. In this subsection, we discuss the sufficient conditions for SBC and SBI systems to converge to an *agreement opinion vector*. In an agreement opinion vector, any two

agents are either disconnected or in consensus. Reaching global consensus, in which all agents hold the same opinion, is an special case of convergence to an agreement opinion vector. Note that it is possible for an SBC or SBI system to reach an opinion vector that contains two neighbor agents with separate opinions in finite time. For instance, the trajectory of an SBC system with initial opinion vector $[0 \;\; 2 \;\; 3 \;\; 4.5 \;\; 7]^T$ and bounds vector $[0.01 \;\; 3 \;\; 0.01 \;\; 3 \;\; 0.01]^T$ exhibits convergence to a fixed profile in finite time. While, the SBC digraph of the limiting opinion vector contains open-minded agents.

PROPOSITION 4.10 (Properties of agreement opinion vectors). *For any agreement opinion vector $\tilde{y} \in \mathbb{R}^n$ in an SBC or SBI system:*

(i) $\min_{i\in\{1,\dots,n\}} \epsilon_i(\tilde{y}) > 0$, *where $\epsilon(\tilde{y})$ is the equi-topology distance of $\tilde{y}$; and*

(ii) if $\tilde{y}$ is the limiting opinion vector of a trajectory, then the trajectory reaches $\tilde{y}$ in finite time.

Proof. Regarding statement (i), by contradiction assume that $\min_{i\in\{1,\dots,n\}} \epsilon_i(\tilde{y}) = 0$. Then based on equation (4.1), there exist agents i and j such that $|\tilde{y}_i - \tilde{y}_j| = r_i$. The latter equation tells us that $j \in \mathcal{N}_i(\tilde{y})$ in an SBC digraph or $i \in \mathcal{N}_j(\tilde{y})$ in an SBI digraph, while their opinions are different from each other by r_i, which contradicts the definition of agreement opinion vectors. Regarding statement (ii), consider trajectory $x(t)$ that converges to $\tilde{y}$. Then, previous statement shows that the limiting opinion vector of $x(t)$ satisfies the condition of Lemma 4.8. Therefore, there exists time step τ such that the proximity digraphs $G_r(\tilde{y})$ and $G_r(x(t))$ are equal for all $t \geq \tau$. On the other hand, the proximity digraph of an agreement opinion vector contains only closed-minded components. Hence, the agents in each WCC of $G_r(x(\tau))$ reach consensus at the next iteration. □

One sufficient condition that guarantees asymptotic consensus in "agreement algorithms", which includes SBC and SBI systems, is given in [25, Theorem 2.4] and is as follows. Take a trajectory $x(t)$ of an SBC or SBI system with proximity digraph $G_r(x(t)) = \big(V, E(x(t))\big)$, where V and $E(x(t))$ are the sets of nodes and edges of the digraph, respectively. If there exists τ such that the graph $\big(V, E(x(k\tau)) \cup E(x(k\tau + 1)) \cup \cdots \cup E(x((k+1)\tau - 1))\big)$ is strongly connected for all $k \in \mathbb{Z}_{\geq 0}$, then all entries of $x(t)$ converge to one real number. However, this sufficient condition requires knowledge of the system for infinite time, and is the same for both SBC and SBI systems. Hence, we derive sufficient conditions that are required to hold in one time step, and also make it possible to compare SBC and SBI systems in support of Conjecture 2.4. Let us first define the *opinion interval* of any subgraph of an SBC or SBI digraph be a closed interval in $\mathbb{R}$ between that subgraph's minimum and maximum opinions.

PROPOSITION 4.11 (Sufficient conditions for convergence to an agreement opinion vector). *Consider the opinion vector $y \in \mathbb{R}^n$ in an SBC or SBI system with the following properties:*

(i) the opinion intervals of any two WCC's of the proximity digraph are separated from each other by a distance strictly larger than the maximum confidence or influence bounds of the agents in those WCC's; and

(ii) it is true that:

- *for any WCC of y's SBC digraph, with m agents, at least $m-1$ agents have confidence bounds larger than that WCC's opinion interval; and*
- *for any WCC of y's SBI digraph, at least one agent has influence bound larger than that WCC's opinion interval.*

Then, the trajectories of both SBC and SBI systems with the initial opinion vector y converge to agreement opinion vectors in finite time. Moreover, in every WCC of

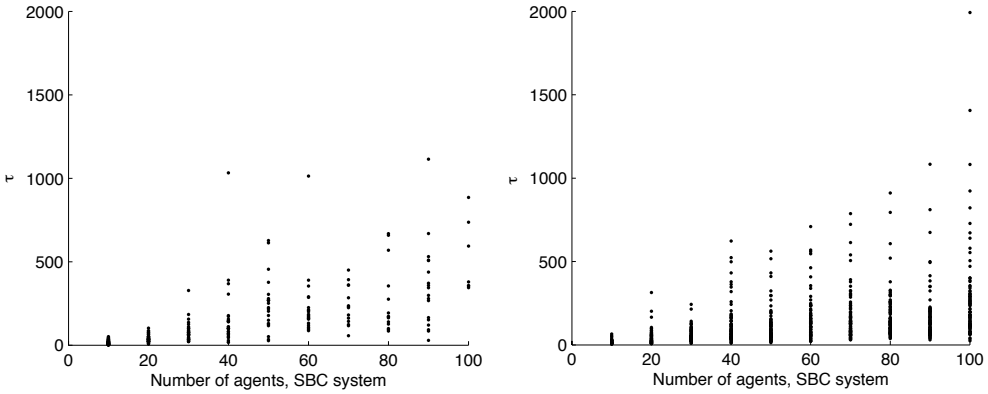

FIG. 5.1. *In one thousand simulations of SBC systems, the time step τ at which the trajectory of each system satisfied the sufficient condition of Theorem 4.4 is plotted versus the number of agents in that system. The left and right plots, respectively, illustrate time τ for trajectories that converged in finite time and infinite time. Each initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. For each agent number hundred simulations are performed. All trajectories satisfied the special case of sufficient condition of Theorem 4.4, stated in Remark 4.7 in finite time.*

either of the SBC or SBI digraphs, at least one node is an out-neighbor of all nodes in that WCC for all $t \geq 0$.

REMARK 4.12. *Any trajectory of an SBC or SBI system that converges to an agreement opinion vector will eventually satisfy the conditions of Proposition 4.11.*

Proof. [Proof of Proposition 4.11] Let us denote either of the SBC or SBI digraphs of y by $G_r(y)$. In an SBC or SBI system, the smallest and largest opinions in a separate WCC of the proximity digraph are, respectively, non-decreasing and non-increasing in one iteration [3]. This fact tells us that for all $t \geq 0$: first, under the condition (i), the two sets of nodes of two separate WCC's in $G_r(x(0))$ remain separate in $G_r(x(t))$; second, if the condition (ii) holds for $G_r(x(0))$, then it also holds for $G_r(x(t))$. Now, under condition (ii) for both SBC and SBI systems, any WCC of $G_r(x(t))$ contains at least one agent that is an out-neighbor of all agents in that WCC for all $t \geq 0$. Denote one such agent in a WCC by s, then that WCC's agents with maximum and minimum opinions update their opinions by taking an average of their out-neighbors, including s. Hence, at the next iteration, their opinions will converge to s's opinion, which results in an strict decrease in the opinion interval of the WCC. Since the confidence or influence bounds are strictly greater than zero, there exists a time step after which the opinion interval of the WCC is larger than the minimum confidence or influence bound. Consequently, all agents become each others out-neighbors, and the WCC becomes one closed-minded component. □

5. Numerical Analysis. In this section, we provide extensive simulation results that demonstrate the results of Section 4 and are consistent with our conjectures. We performed 2000 simulations: 100 simulations of both SBC and SBI systems for ten different agent numbers. In each simulation, the initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. The time steps τ at which trajectories satisfied the condition of Theorem 4.4 are plotted in Figures 5.1 and 5.2. All the 2000 SBC and SBI trajectories eventually satisfied the special case of the sufficient condition of Theorem 4.4, stated in Re-

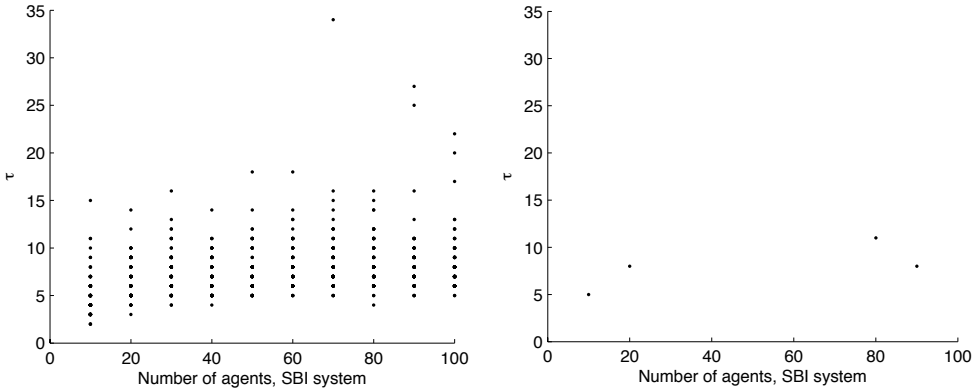

FIG. 5.2. *In one thousand simulations of SBI systems, the time step τ at which the trajectory of each system satisfied the sufficient condition of Theorem 4.4 is plotted versus the number of agents in that system. The left and right plots, respectively, illustrate time τ for trajectories that converged in finite time and infinite time. As shown, only four SBI trajectories converged in infinite time. Each initial opinion vector and bounds vector are generated randomly and uniformly distributed on [0, 1] and [0, 0.3], respectively. For each agent number hundred simulations are performed. All trajectories satisfied the special case of sufficient condition of Theorem 4.4, stated in Remark 4.7 in finite time.*

mark 4.7. In other words, for each trajectory $x(t)$, there exists time τ such that $x(\tau)$ belongs to the invariant equi-topology neighborhood of its own final value at constant topology $\text{fvct}(x(\tau))$. Thus, $\text{fvct}(x(\tau))$ is an equilibrium opinion vector, and is equal to the limiting opinion vector of $x(t)$. The frequency of occurrence of this special case is intuitively explained by the following statements: First, by Conjecture 2.1, for each trajectory a limiting opinion vector x_∞ exists. Second, for any randomly generated opinion vector y and bounds vector r, the probability of having $\min_{i\in\{1,\dots,n\}} \epsilon_i(y) = 0$, where $\epsilon(y)$ is the equi-topology distance of y, is equal to zero, and one can assume that the same holds for any limiting opinion vector. Third, Lemma 4.8 tells us that if the limiting opinion vector satisfies $\min_{i\in\{1,\dots,n\}} \epsilon_i(x_\infty) > 0$, then the trajectory eventually satisfies the mentioned special case of condition of Theorem 4.4.

In above mentioned simulations, for each agent number, the percentage of SBC and SBI trajectories that reached a fixed profile in finite time are plotted in Figure 5.3. Clearly, Figure 5.3 supports Conjecture 2.4. To explain this frequency of convergence of SBI trajectories in finite time as compared with SBC trajectories, we use the results of Subsection 4.2. For uniformly randomly generated opinion vector and bounds vector, an SBI digraph is more likely to satisfy condition (i) of Proposition 4.11 than an SBC digraph. One can assume that the same holds for a trajectory with uniformly randomly generated initial opinion vector and bounds vector.

In the next section, based on our "constant topology in finite time" conjecture, we assume that the interconnection topology in an SBC or SBI system remains constant for infinite time and address the following questions: How the three classes of agents behave? How groups of agents affect each other? And can one explain the observed pseudo-stable behavior of trajectories, as stated in Conjecture 2.2?

6. The Rate and Direction of Convergence under Fixed Topology. In this section, we analyze the rates and directions of convergence of separate classes of agents in the SBC and SBI systems under fixed interconnection topology as time

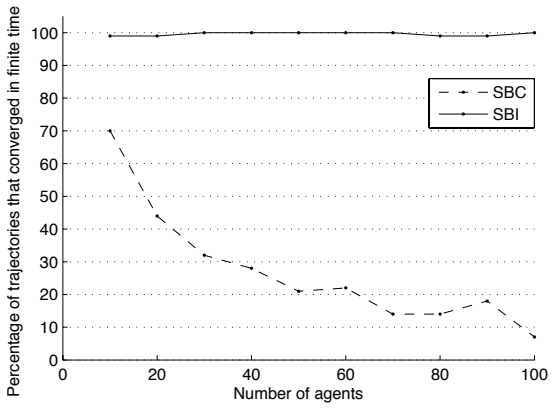

FIG. 5.3. *For each agent number 100 SBC systems and 100 SBI systems are simulated, as explained in Figures 5.1 and 5.2. The percentage of SBC (blue) and SBI (green) trajectories that reached their limiting opinion vector, which is an agreement opinion vector, in finite time is plotted versus agent number.*

goes to infinity. This analysis proves that the system shows a pseudo-stable behavior under fixed topology.

DEFINITION 6.1 (Agent's per-step convergence factor). *In an SBC or SBI system with trajectory* $x(t) \in \mathbb{R}^n$*, we define the* per-step convergence factor *of an agent* i *whose* $x_i(t) - \mathrm{fvct}_i(x(t))$ *is nonzero to be*

$$k_i(x(t)) = \frac{x_i(t+1) - \mathrm{fvct}_i(x(t))}{x_i(t) - \mathrm{fvct}_i(x(t))}.$$

The per-step convergence factor of a network of agents was previously introduced in [34] to measure the overall speed of convergence toward consensus.

REMARK 6.2. *Consider a converging trajectory* $x(t)$ *whose limiting opinion vector is equal to* $\mathrm{fvct}(x(t))$ *for all* $t \geq 0$*. Then,* $x(t)$ *exhibits a pseudo-stable behavior, see equation (2.2), if and only if for all* $i \in \{1, \ldots, n\}$

$$\begin{cases} 0 < k_i(x(t)) < 1, & \text{if } k_i(x(t)) \text{ exists,} \\ x_i(t) = x_i(t+1) = \mathrm{fvct}_i(x(t)), & \text{otherwise.} \end{cases}$$

DEFINITION 6.3 (Leader SCC). *For opinion vector* $y \in \mathbb{R}^n$*, let* $G_r(y)$ *denote either its SBC or SBI digraph. Consider an SBC or SBI system with opinion vector* $y \in \mathbb{R}^n$*. For any open-minded SCC,* $S_k(y)$*, of* $G_r(y)$*, denote the set of its open-minded successor SCC's, including* $S_k(y)$*, by* $\mathcal{M}(S_k(y))$*. We define* $S_k(y)$*'s* leader SCC *to be the SCC whose adjacency matrix has the largest spectral radius among all SCC's of* $\mathcal{M}(S_k(y))$*.*

Note that in SBC and SBI digraphs, the adjacency matrix of a large SCC has a large spectral radius, hence that SCC tends to become a leader SCC for its predecessors.

THEOREM 6.4 (Evolution under constant topology). *Consider an SBC or SBI system, denote its trajectory by* $x(t)$ *and proximity digraph by* $G_r(x(t))$*. Assume that there exists a time* τ *after which* $G_r(x(t))$ *remains unchanged, that is,* $G_r(x(t)) = G_r(x(\tau))$*. Then, the following statements hold:*

(i) $\text{fvct}(x(t)) = \text{fvct}(x(\tau))$ *for all* $t \geq \tau$.

(ii) $G_r(x(\tau))$ *contains no moderate-minded component.*

(iii) Consider any open-minded SCC $S_k(x(t))$ *of* $G_r(x(t))$ *and its leader SCC* $S_m(x(t))$*, with adjacency matrices denoted by* Θ_k *and* Θ_m*, respectively. Then,*

(a) for any $i \in S_k(x(t))$*, either* $x_i(t) - \text{fvct}_i(x(t)) = 0$ *or its per-step convergence factor converges to the spectral radius of* Θ_m *as time goes to infinity, and*

(b) if the spectral radius of Θ_k *is strictly less than that of* Θ_m*, then there exists* $t_1 \geq \tau$ *such that for all* $i \in S_k(x(t))$*,* $j \in S_m(x(t))$*, and* $t \geq t_1$*,*

$$\begin{aligned} x_j(t_1) < \text{fvct}_j(x(t_1)) \quad &\implies \quad x_i(t) \leq \text{fvct}_i(x(t)), \\ x_j(t_1) > \text{fvct}_j(x(t_1)) \quad &\implies \quad x_i(t) \geq \text{fvct}_i(x(t)). \end{aligned}$$

(iv) There exists time $t_2 \geq \tau$ *such that for all* $t \geq t_2$*,* $x(t)$ *exhibits a pseudo-stable behavior, see equation* (2.2).

REMARK 6.5 (Interpretation of statement (iii) in Theorem 6.4). *Parts (a) and (b) tell us, respectively, that the rates and directions of convergence of opinions in an open-minded SCC toward the final value at constant topology are governed by the direction and rate of convergence of its leader SCC. It is easy to that the per-step convergence factor has an inverse relation with the rate of convergence to the final value at constant topology. Therefore, Theorem 6.4 implies that under fixed interconnection topology, individuals converge to a final decision as slow as the slowest group of agents whom they listen to.*

Proof. Statement (i) is a direct consequence of $A(x(t)) = A(x(\tau))$ for all $t \geq \tau$. Statement (ii) can be proved similar to part (ii)b of Proposition 3.2. It was shown that under fixed interconnection topology, all agents in one moderate-minded SCC of an SBC or SBI digraph reach consensus as time goes to infinity. Since, the bounds vector is strictly greater than zero, there exists a time step after which the adjacency matrix of one moderate-minded SCC transforms into a complete consensus matrix, which contradicts the assumption of having fixed topology for infinite time. Before proving statement (iii), since the canonical permutation matrix remains unchanged, let us assume that the opinions in $x(\tau)$ are ordered such that $A(x(\tau)) = \overline{A}(x(\tau))$. Furthermore, owing to the fixed interconnection topology, we drop the $x(t)$ argument for simplicity. Therefore, by equation (2.3),

$$A(x(t)) = \begin{bmatrix} C & 0 \\ \Theta_C & \Theta \end{bmatrix}.$$

Now, for all $t > \tau$ we have

$$x(t) - \text{fvct}(x(\tau)) = \begin{bmatrix} Cx_C(\tau) \\ x_\Theta(t) \end{bmatrix} - \begin{bmatrix} Cx_C(\tau) \\ \text{fvct}_\Theta(x(\tau)) \end{bmatrix} = \begin{bmatrix} 0 \\ x_\Theta(t) - \text{fvct}_\Theta(x(\tau)) \end{bmatrix},$$

where $x_C(t)$ and $x_\Theta(t)$ are the opinion vectors of agents in closed- and open-minded classes respectively. Using $\text{fvct}(x(\tau)) = A(x(\tau))\,\text{fvct}(x(\tau))$, the following recurrence relation holds

$$x_\Theta(t+1) - \text{fvct}_\Theta(x(\tau)) = \Theta(x_\Theta(t) - \text{fvct}_\Theta(x(\tau))) \quad \forall\ t \geq \tau. \tag{6.1}$$

Consider an open-minded WCC of $G_r(x(t))$, denoted by W_1. Let Θ_1 denote W_1's adjacency matrix, and $x_1(t)$ denote the trajectory of nodes of W_1. Under fixed interconnection topology, the trajectory of each WCC is independent of others, thus for

all $t \geq 0$, $x_1(t+\tau) - \text{fvct}_1(x(\tau)) = \Theta_1^t(x_1(\tau) - \text{fvct}_1(x(\tau)))$. According to the block lower triangular from of Θ_1,

$$\Theta_1^t = \begin{bmatrix} \Theta_{11}^t & & 0 \\ \Theta_{21}^{(t)} & \Theta_{22}^t & \\ \vdots & & \ddots \end{bmatrix},$$

where each Θ_{ii} is the adjacency matrix of an SCC, denoted by S_{ii}, of W_1. Let $x_{ii}(t)$ be the opinion trajectory of nodes of S_{ii}. Clearly, S_{11} is one of the sink SCC's in W_1, and Θ_{ii}'s are ordered in Θ_1 according to the distance of S_{ii}'s to the sinks. For simplicity, we prove statement (iii) for S_{11} and an SCC that is the direct predecessor of S_{11}. Without loss of generality, let S_{22} be one such SCC. The proof for the rest of open-minded SCC's is similar.

Each block Θ_{ii} is nonnegative and primitive. By Perron-Frobenius Theorem: the spectral radius of Θ_{ii}, denoted by λ_i, is positive and a simple eigenvalue of Θ_{ii}; and there exists a positive eigenvector ν_i for Θ_{ii} associated to λ_i. Any Θ_{ii} can be written in Jordan normal form by some similarity transformation

$$\Theta_{ii} = QJQ^{-1} = \left[\begin{array}{c|c} \nu_i & Q_e \end{array}\right] \begin{bmatrix} \lambda_i & \mathbf{0} \\ \mathbf{0} & J_e \end{bmatrix} \begin{bmatrix} w_i \\ \hline Q_e^{(-1)} \end{bmatrix},$$

where w_i is the first row of Q^{-1}. Consequently,

$$\lim_{t\to\infty} \Theta_{ii}^t = \lim_{t\to\infty} (\lambda_i^t \nu_i w_i + Q_e J_e^t Q_e^{(-1)}) = \lim_{t\to\infty} \lambda_i^t \nu_i w_i. \tag{6.2}$$

In any open-minded WCC, the sink SCC is not effected by other SCC's, hence a sink SCC is its own leader. Therefore, for all $t \geq 0$,

$$x_{11}(t+\tau) - \text{fvct}_{11}(x(\tau)) = \Theta_{11}^t(x_{11}(\tau) - \text{fvct}_{11}(x(\tau))).$$

In the interest of simplicity, let us denote the vector $x_{ii}(t) - \text{fvct}_{ii}(x(\tau))$ by $\Delta_{ii}(t)$, then we have

$$\lim_{t\to\infty} \Delta_{11}(t) = \nu_1 w_1 \Delta_{11}(\tau) \lim_{t\to\infty} \lambda_1^t. \tag{6.3}$$

Regarding part (iii)a for S_{11}, for the per-step convergence factor of any $i \in S_{11}$ we have

$$\lim_{t\to\infty} k_i(x(t)) = \lim_{t\to\infty} \frac{\lambda_1^{t+1} w_1 \Delta_{11}(\tau) \nu_{1i}}{\lambda_1^t w_1 \Delta_{11}(\tau) \nu_{1i}} = \lambda_1.$$

Regarding part (iii)b for S_{11}, since $\lambda_1^t w_1 \Delta_{11}(\tau)$ is a scalar, λ_1 is positive, and ν_1 is a positive vector, all entries of vector $\nu_1 w_1 \Delta_{11}(\tau) \lambda_1^t$ have the same sign. Therefore, there exists time $T \geq \tau$ after which all entries of $\Delta_{11}(t)$ have the same sign for all $t \geq T$.

Here, we prove the two statements for S_{22}. It can be computed that for all $t \geq 0$

$$\Delta_{22}(t+\tau) = \sum_{i=0}^{t-1} \Theta_{22}^i \Theta_{21} \Theta_{11}^{t-i-1} \Delta_{11}(\tau) + \Theta_{22}^t \Delta_{22}(\tau).$$

Now, to find $\lim_{t\to\infty} \Delta_{22}(t)$, we consider three cases:
1) If $\lambda_1 > \lambda_2$, then S_{11} is S_{22}'s leader. According to the transient analysis of the

reducible Markov chains from [11, Section 5.6], and granted that λ_1 and λ_2 are strictly less than one, see Section 2.2,

$$\lim_{t\to\infty}\sum_{i=0}^{t-1}\Theta_{22}^{i}\Theta_{21}\Theta_{11}^{t-i-1} = \big(\lim_{t\to\infty}\sum_{i=0}^{t-1}\Theta_{22}^{i}\big)\Theta_{21}\lim_{t\to\infty}\Theta_{11}^{t}$$
$$= (I-\Theta_{22})^{-1}\Theta_{21}\lim_{t\to\infty}\lambda_1^t\nu_1 w_1.$$

Therefore,

$$\lim_{t\to\infty}\Delta_{22}(t) = (I-\Theta_{22})^{-1}\Theta_{21}\nu_1 w_1\Delta_{11}(\tau)\lim_{t\to\infty}\lambda_1^t. \tag{6.4}$$

Regarding part (iii)a for S_{22}, for any $i\in S_{22}$ we have

$$\lim_{t\to\infty}k_i(x(t)) = \lim_{t\to\infty}\frac{\lambda_1^{t+1}w_1\Delta_{11}(\tau)[(I-\Theta_{22})^{-1}\Theta_{21}\nu_1]_i}{\lambda_1^{t}w_1\Delta_{11}(\tau)[(I-\Theta_{22})^{-1}\Theta_{21}\nu_1]_i} = \lambda_1.$$

Regarding part (iii)b for S_{22}, since $(I-\Theta_{22})^{-1}\Theta_{21}\nu_1$ is a nonnegative vector, all entries of the vector on the right hand side of equations (6.3) and (6.4) have the same sign as the scalar $w_1\Delta_{11}(\tau)$.
2) If $\lambda_1<\lambda_2$, then S_{22} is its own leader. Similarly,

$$\lim_{t\to\infty}\Delta_{22}(t) = w_2\big(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau)+\Delta_{22}(\tau)\big)\nu_2\lim_{t\to\infty}\lambda_2^t, \tag{6.5}$$

where $w_2\big(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau)+\Delta_{22}(\tau)\big)$ is a scalar, λ_2 is positive, and ν_2 is a positive vector. Regarding part (iii)a for S_{22}, for any $i\in S_{22}$ we have

$$\lim_{t\to\infty}k_i(x(t)) = \lim_{t\to\infty}\frac{\lambda_2^{t+1}w_2(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau)+\Delta_{22}(\tau))\nu_{2i}}{\lambda_2^{t}w_2(\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau)+\Delta_{22}(\tau))\nu_{2i}} = \lambda_2.$$

Regarding part (iii)b for S_{22}, all entries of the vector on the right hand side of equation (6.5) have the same sign.
3) if $\lambda_1=\lambda_2=\lambda$, then we have

$$\lim_{t\to\infty}\Delta_{22}(t) = \big(\alpha\nu_2+\beta(I-\Theta_{22})^{-1}\Theta_{21}\nu_1\big)\lim_{t\to\infty}\lambda^t,$$

where $\beta = w_1\Delta_{11}(\tau)$ and $\alpha = w_2\Theta_{21}(I-\Theta_{11})^{-1}\Delta_{11}(\tau)+w_2\Delta_{22}(\tau)$. Regarding part (iii)a for S_{22}, for any $i\in S_{22}$ we have

$$\lim_{t\to\infty}k_i(x(t)) = \lim_{t\to\infty}\frac{\lambda^{t+1}(\alpha\nu_{2i}+\beta[(I-\Theta_{22})^{-1}\Theta_{21}\nu_1]_i)}{\lambda^{t}(\alpha\nu_{2i}+\beta[(I-\Theta_{22})^{-1}\Theta_{21}\nu_1]_i)} = \lambda.$$

Regarding part (iii)b for S_{22}, notice that the theorem does not discuss the case with equal spectral radii.

Finally, statement (iv) is proved utilizing previous statements. For any $i\in G_r(x(t))$ two cases exists. First, if i belongs to a closed-minded SCC, then $x_i(t) = x_i(\tau+1)$ for all $t>\tau$, and hence $k_i(x(t))$ does not exist. Second, if i belongs to an open-minded SCC $S_k(x(t))$, then according to part (iii)a, either $x_i(t) = \text{fvct}_i(t)$ or $k_i(x(t))$ converges to the spectral radius of the adjacency matrix of $S_k(x(t))$'s leader SCC. This spectral radius is proved in Section 2.2 to be strictly larger than zero and strictly smaller than one. In other words, there exists time t_2 such that for all $t\geq t_2$, $0<k_i(x(t))<1$. Therefore, according to Remark 6.2, $x(t)$ exhibits pseudo-stable behavior. □

In Figures 6.1 and 6.2, we provide numerical examples to facilitate the understanding of the conditions and results of Theorem 6.4.

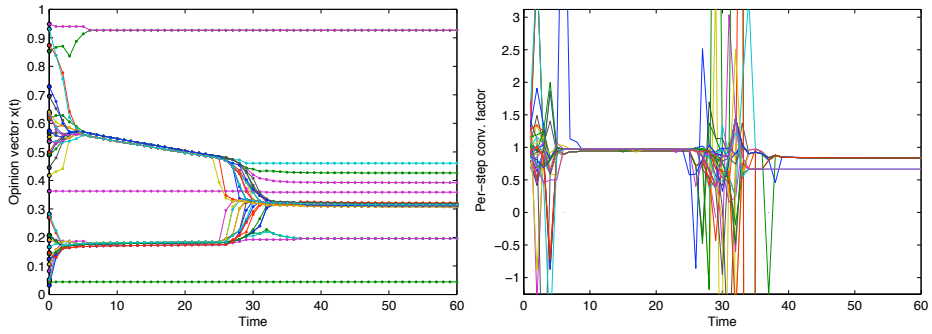

FIG. 6.1. *The trajectory of an SBC system (left) and the per-step convergence factor of its open-minded agents (right) are illustrated. The initial opinion vector and confidence bounds vector are generated randomly. This system satisfies the condition of Theorem 4.4 at* $t = 50$. *Moreover, since* $x(50) \in \mathcal{B}_{\text{iet}}(\text{fvct}(x(50)))$, *the SBC digraph* $G_r(x(t))$ *is equal to* $G_r(\text{fvct}(x(50)))$ *for all* $t \geq 50$. *The digraph* $G_r(\text{fvct}(x(50)))$ *contains two open-minded SCC's, denoted by* S_1 *and* S_2, *while* S_2 *is a predecessor of* S_1. *The spectral radii of the adjacency matrices of* S_1 *and* S_2 *are equal to 0.6667 and 0.8381, respectively. Therefore, both* S_1 *and* S_2 *are their own leader SCC's, and by Theorem 6.4 the per-step convergence factors of their agents converge to 0.6667 and 0.8381, respectively. The right plot verifies that the per-step convergence factors of all open-minded agents converge to those two values.*

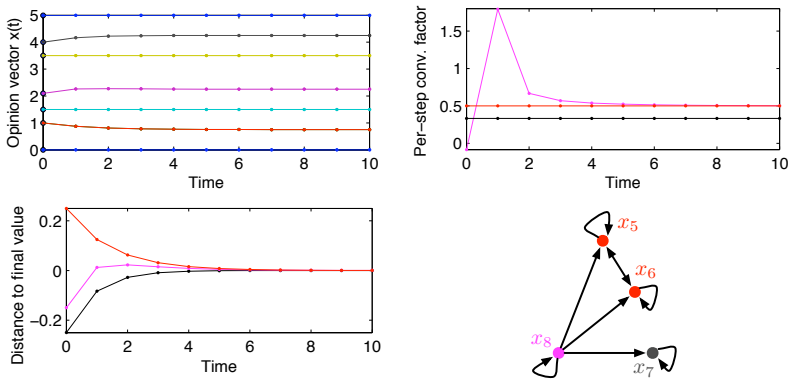

FIG. 6.2. *An SBC trajectory* $x(t)$ *is plotted on the top left, the open-minded agents per-step convergence factors on the top right, the open-minded agents distances to their final values at constant topology* $x_i(t) - x_i^*(x(t))$ *on the bottom left, and the open-minded subgraph of* $G_r(x(t))$ *is illustrated on the bottom right. This system is simulated with the initial vector* $x(0) = [0\ 1.5\ 3.5\ 5\ 1\ 1\ 4\ 2.1]^T$ *and confidence bounds* $r = [0.01\ 0.01\ 0.01\ 0.01\ 1\ 1\ 1\ 3]^T$. *For all* $t \geq 0$, *the SBC digraph* $G_r(x(t))$ *remains unchanged and contains three open-minded SCC's:* $\{x_5, x_6\}$, $\{x_7\}$, *and* $\{x_8\}$. *The spectral radii of the adjacency matrices of these SCC's are* 0.5, 0.333, *and* 0.125, *respectively. The two SCC's* $\{x_5, x_6\}$ *and* $\{x_7\}$ *are successors of* $\{x_8\}$, *and based on their spectral radii,* $\{x_5, x_6\}$ *is* $\{x_8\}$*'s leader SCC. We can see that the per-step convergence factor of* x_8 *converges to* 0.5. *Furthermore, the sing of its direction of convergence toward the final value, i.e, the sign of* $x_8(t) - x_8^*$, *is the same as the leader's after* $t = 1$. *These facts support Theorem 6.4.*

7. Conclusion and Future Work. This paper introduced a synchronized bounded influence (SBI) model of opinion dynamics, which is similar to the heterogeneous bounded confidence model introduced by Hegselmann and Krause, which we called synchronized bounded confidence (SBC) model. First, we conjectured that in both SBC and SBI systems, for each trajectory there exists a finite time, after which the topology of the interconnection network remains unchanged, hence, the trajectory

converges to a limiting opinion vector. Second, we conjectured that if a trajectory does not reach a fixed profile in finite time, then it eventually shows a pseudo-stable behavior. We partly proved our first conjecture, and the second conjecture is proved assuming that the first one is true. We designed a classification of agents that is employed in computing the equilibria of the system. We introduced the equi-topology neighborhood and the invariant equi-topology neighborhood of the equilibria of the system. Based on these neighborhoods, we derived sufficient condition for both SBC and SBI systems to guarantee that the interconnection topology remains unchanged for infinite time in a trajectory, and therefore, the trajectory converges to a steady state. In our simulation results, it is observed that for uniformly randomly generated initial opinion vector and bounds vector, the trajectories of both systems eventually satisfy the mentioned sufficient condition with probability one. However, the eventual convergence of every trajectory of the SBC and SBI systems to a steady state is still an open problem. Third, we conjectured that, for uniformly randomly generated initial opinion vector and bounds vector, the simulations of SBI systems converge in fewer time steps and more often in finite time than SBC systems. We derived sufficient conditions for convergence in finite time for SBC and SBI systems separately that intuitively explains our third conjecture. Finally, we studied the trajectory of both SBC and SBI systems when they update their opinions under fixed interconnection topology for infinite time. We showed the existence of a leader group for each group of agents that determines the follower's rate and direction of convergence.

The main future challenge is to prove that all SBC and SBI systems converge to steady states. One approach is to prove that in each system, any trajectory is eventually confined to the invariant equi-topology neighborhood of an equilibrium opinion vector of the system. Moreover, the fact that the SBI systems converge in finite time more often than the SBC systems might be explained by a probability analysis on the topology of proximity digraphs.

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