Matrix Concentration & Applications to Combinatorics

Khintchine/Bernstein's Inequality

$\varepsilon_1, \ldots, \varepsilon_n \sim \{\pm 1\}$ uniform indep $\quad \sigma = \sqrt{\Sigma_i a_i^2}$

$a_1 \text{---} a_n$: arbitrary reals.

std-dev

Then $\Pr\left[\left| \Sigma_i \varepsilon_i a_i \right| \geq t \cdot \sigma \right]$

$$\leq 2 \cdot e^{-\frac{t^2}{2}}$$

define operator norm here

Non-Commutative/ Matrix Khintchine

$\varepsilon_1 \text{---} \varepsilon_n \sim$ iid Rademachers

$A_1 \text{---} A_n$: arbitrary $n \times n$ symm. matrices.

[Lust Piquard, Pisier '91]

$\sigma = \| \Sigma_i A_i^2 \|_2$

$$\Pr\left[\left\|\sum_i \varepsilon_i A_i\right\|_2 \geq t\cdot\sigma\right]$$

$$\leq 2\cdot n\cdot e^{-\frac{t^2}{2}}$$

$$\mathbb{E}\left[\left\|\sum_i \varepsilon_i A_i\right\|_2\right] \leq \sqrt{2\log 2n}\cdot\sigma$$

Very useful inequality! In these 3 lectures, I want to tell you 3 applications of it to combinatorial facts.

controls expected maximum

$$\mathbb{E}\max_{\substack{u,v\\ \text{unit}}}\left[u^T\left(\sum_i \varepsilon_i A_i\right)v\right] \lesssim O(\sqrt{\log n})\,\sigma$$

Won't prove it but want to tell you a few applications so that you can add it to your toolkit

The applications I chose are in combinatorics.

Graph Sparsification:

<u>Def</u>: $G'(V, E')$ is an ε-cut sparsifier for $G(V, E)$ if

$$\forall S \subseteq V, \quad |E_{G'}(S, \bar{S})| \in (1 \pm \varepsilon) |E_G(S, \bar{S})|$$

<u>Thm</u>: every G admits an ε-cut sparsifier with $O\left(\frac{n}{\varepsilon^2} \cdot \log n\right)$ edges.

Lap solvers
expander graphs

$O\left(\frac{n}{\varepsilon^2}\right)$!

Batson-Spielman-Srivastava

Girth problems [Bollobas'78]

[Alon-Hoory-Linial'02]
Every G with avg deg d has a cycle of length $\leq \lceil 2 \cdot \log_{d-1}(n) \rceil$.

Hypergraph Generalization [Feige] → lots of apps

Codes (Local Codes)

$E : \overset{b}{\mathbb{F}_2^k} \to \overset{x, y}{\mathbb{F}_2^n}$, q-LDC if

given an $i \in [k]$, y corr. codewords

decode b_i by reading 3 bits of y.

what's the smallest $n = n(k)$ for which such codes exist?

Why?

1. beautiful combinatorial result
2. generalizes expander graphs

such that G' approximates G

Cut sparsifier

$\forall\, S \subseteq V$, $|E_{G'}(S,\bar{S})| \in (1 \pm \varepsilon)\, |E_G(S,\bar{S})|$

Laplacian

$$L_G = D - A_G$$

$$L_e = \begin{array}{c} \\ i \\ \\ j \end{array} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} - \begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$$

$e = \{i,j\}$

has a zero eigenvalue

$$L_G = \sum_{e \in E} L_e, \quad L_{G'} = \sum_{e \in E'} w_e \cdot L_e$$

$$x^T L_G x = \sum_{e \in E} x^T L_e x$$

$$= \sum_{e \in E} x_i^2 + x_j^2 - 2x_i x_j$$

$$= \sum_{e \in E} (x_i - x_j)^2$$

So if $x \in \{\pm 1\}^n$, $x^T L_G x$

$$= 4 \cdot \mathrm{cut}(x)$$

Thus: cut sparsification

$$\Updownarrow$$

$$x^T L_{G'} x \in (1 \pm \varepsilon) \cdot x^T L_G x$$

$$\cancel{\forall x \in \{\pm 1\}^n}$$

Spectral sparsification: $\forall x$.

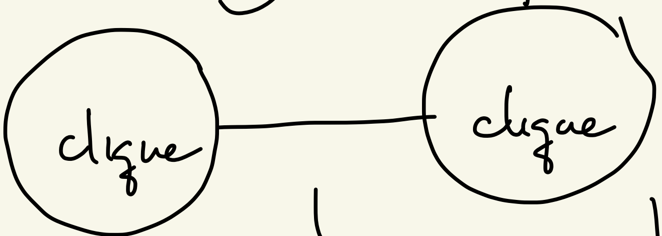
[Karger'93, Benczur-Karger'95]

$\forall G$, $\exists G'$ with $O(\frac{n}{\varepsilon^2} \cdot \log n)$ edges

s.t. G' is a ε-cut sparsifier of G.

Idea: "importance" sampling

Sample each edge independently



must sample this edge!

$$L_G = \sum_{e \in E} L_e$$

Need: $$x^T\left(\sum_{e \in E'} w_e \cdot L_e\right)x$$

$$= (1 \pm \varepsilon)\, x^T \sum_{e \in E} L_e x$$

relative error

Key Idea

$$\tilde{L}_e = L_G^{-1/2} \cdot L_e \cdot L_G^{-1/2}$$

Note: $\sum \tilde{L}_e = \mathbb{I}_\perp$

$$\Big\| \sum_{e \in E'} w_e \cdot \tilde{L}_e - \mathbb{I}_\perp \Big\|_2 \leq \varepsilon.$$

$\forall$ unit u

$$\Leftrightarrow \quad u^T \Big(\sum_{E'} w_e \cdot \tilde{L}_e \Big) u \in (1 \pm \varepsilon) \cdot \|u\|_2^2$$

Plug in $u = L_G^{1/2} \cdot u'$

$$\Leftrightarrow \quad u'^T \Big(\sum_{E'} w_e \cdot L_e \Big) u' \in (1 \pm \varepsilon) \cdot u'^T L_G u'$$

So: goal "sample" $E' \subseteq E$ s.t.

$$\left\| \sum_{e \in E'} w_e \cdot \tilde{L}_e - \mathbb{I}_t \right\|_2 \leq \varepsilon.$$

Iterative halving strategy

0: $E_0 = E$

1. Take a uniformly random sign $s \in \{\pm 1\}^m$

2. $E_{i+1} = \{e \mid e \in E_i, s_e = +1\}$.

$$w_{i+1}(e) = \begin{cases} 2 \cdot w_i(e) & \text{if } e \in E_{i+1} \\ 0 & \text{o/w} \end{cases}$$

Error: $\left\| \sum_{e \in E_1} \tilde{L}_e \cdot 2 - \sum_{e \in E_0} \tilde{L}_e \right\|_2$

$= \left\| \sum_{e \in E_0} s_e \cdot \tilde{L}_e \right\|_2$

Matrix Khintchene

A_1 ———— A_N, $n \times n$, symmetric deterministic

$$\mathbb{E}\left\|\sum b_i A_i\right\|_2 \le \sigma\sqrt{\log n}$$

for $\sigma^2 = \left\|\sum_i A_i^2\right\|_2$

Matrix Analysis Facts:

1. If A is PSD, then so is A^2

2. If $\|A\|_2 = \rho$, then $A \preceq \rho \cdot \mathbb{I}$.

3. $A^2 \preceq (\rho \cdot \mathbb{I}) A$

Pf: $\sum \lambda_i^2 \cdot v_i v_i^T \preceq \sum \rho \cdot \lambda_i v_i v_i^T$.

<u>Lemma:</u> Suppose $\|\tilde{L}_e\|_2 \leq \rho \ \forall e$.

Then, $\left\| \sum_{e \in E_i} s_e \cdot \tilde{L}_e \right\|_2$

$$\lesssim \left\| \sum_e \tilde{L}_e^2 \right\|_2^{1/2} \cdot \sqrt{\log n}$$

$$\lesssim \rho^{1/2} \cdot \sqrt{\log n}$$

$\sum_e \tilde{L}_e \cdot \tilde{L}_e$
$\preceq \sum_e (\rho \cdot \mathbb{I}) \cdot \tilde{L}_e$
$= \rho \cdot \mathbb{I}_\perp$
So $\|\sum_e \tilde{L}_e^2\|_2 \leq \rho$

How small can ρ be?

$$\sum_{e \in E} \|\tilde{L}_e\|_2 = \sum_{e \in E} \mathrm{Tr}(\tilde{L}_e)$$

rank 1

$$= \mathrm{Tr}(\mathbb{I}_\perp)$$

$$= n-1$$

So "best value": $\rho \sim \frac{n-1}{m}$

In best case scenario, error:

$$C \cdot \sqrt{\log n} \left[\sqrt{\frac{n}{m}} + \sqrt{\frac{n}{m/2}} + \dots + \sqrt{\frac{n}{m_{final}}} \right]$$

$$\lesssim O\left(\sqrt{\frac{n}{m_{final}}}\right) \cdot \sqrt{\log n}$$

$$\lesssim \varepsilon \text{ if } m_{final} \geq O\left(\frac{n}{\varepsilon^2} \cdot \log(n)\right)$$

<u>So</u> only remaining issue:

what if some ρ is large?

<u>Idea:</u> 1. Sort edges in ascending order of lev scores.

2. Apply the sign & halve trick to first half

<u>Obs^n</u>: e_1 —— e_m

$$\rho_{\frac{m}{2}} \leq \binom{2n}{m}.$$

<u>Pf</u>: $\rho \cdot \frac{m}{2} \leq n$!

<u>HW</u>: work out the above argument.

[Spielman-Teng]

Today, an iterative construction

Lemma 1: $\sum_e \| L_G^{-1/2} \cdot L_e \cdot L_G^{-1/2} \|_2$
$$= \sum_e \mathrm{Tr}(L_G^{-1/2} \cdot L_e \cdot L_G^{-1/2})$$
$$= \mathrm{Tr}(\sum_e \cdot L_G^{-1/2} \cdot L_e L_G^{-1/2})$$
$$= \mathrm{Tr}(\mathbb{I}_\perp) = n-1.$$

Lemma 2: Sort the edges of G in ascending order of $\| L_G^{-1/2} \cdot L_e \cdot L_G^{-1/2} \|_2$

$e_1, \underset{\frac{n}{2}}{\rule{3cm}{0.4pt}}, e_m.$

Then $e_{\frac{n}{2}}$ has lev score $\leq \left(\frac{2n}{m}\right)$.

Lemma 3:

Suppose max lev score $\lesssim \rho$.

Then, for uniform $s \in \{\pm 1\}^m$

$$\mathbb{E} \left\| \sum s_e \cdot \tilde{L}_e \right\|_2$$

$$\lesssim \sqrt{\log n} \cdot \left\| \sum_e \tilde{L}_e^2 \right\|_2^{1/2}$$

$$= \sqrt{\log n} \cdot \rho$$

Note:

$$\left\| \sum_e \tilde{L}_e^2 \right\|_2 \leq \max_e \|\tilde{L}_e\|_2 \cdot \left\| \sum_e \tilde{L}_e \right\|_2$$

$$= \rho.$$