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4. GRAPHS AND DIGRAPHS I

- *introduction*
- *graph representation*
- *depth-first search*
- *path finding*
- *undirected graphs*

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4. Graphs and Digraphs I

- *introduction*
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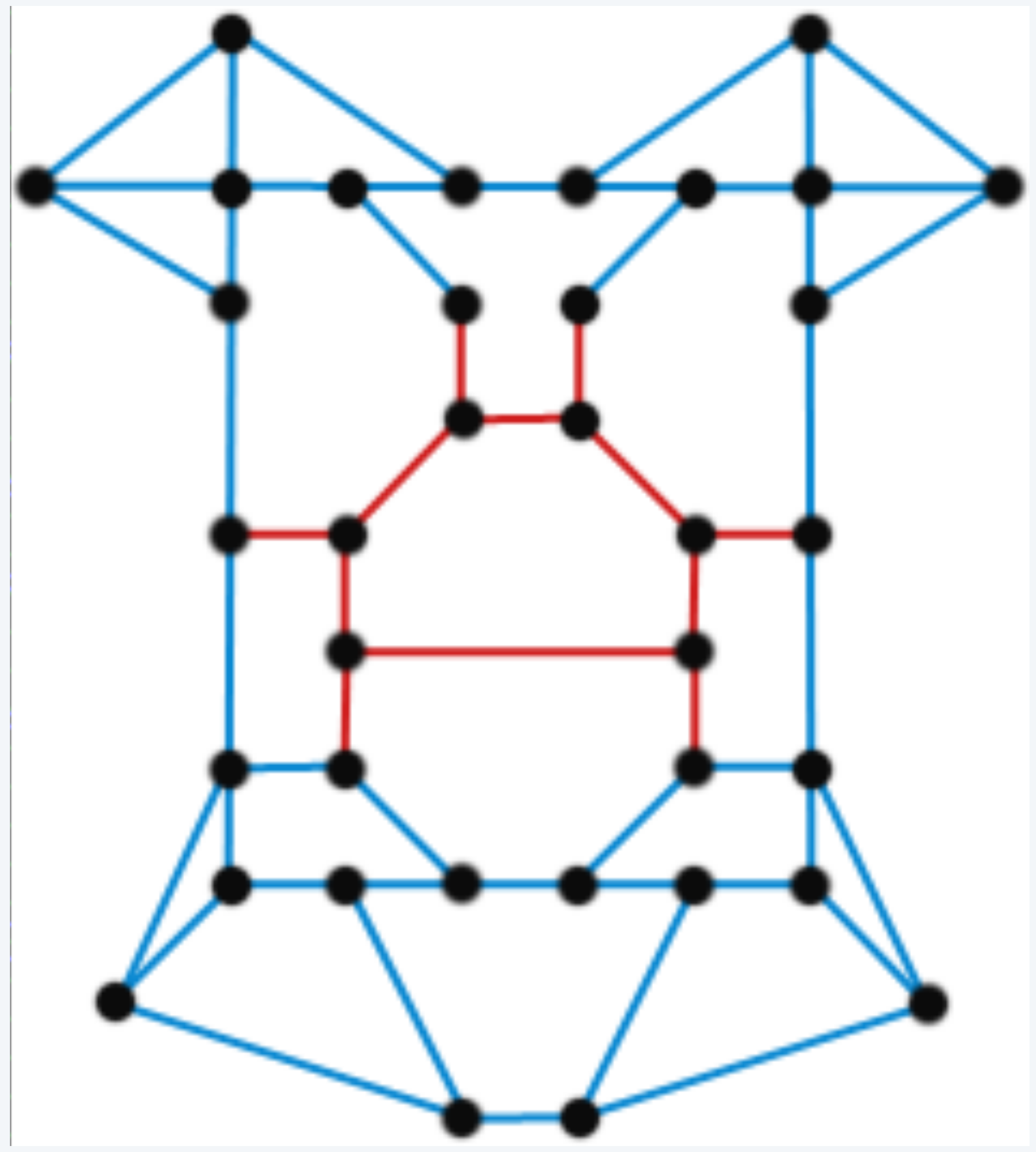
- *depth-first search*
- *path finding*
- *undirected graphs*

Graphs

Graph. Set of vertices connected pairwise by edges.

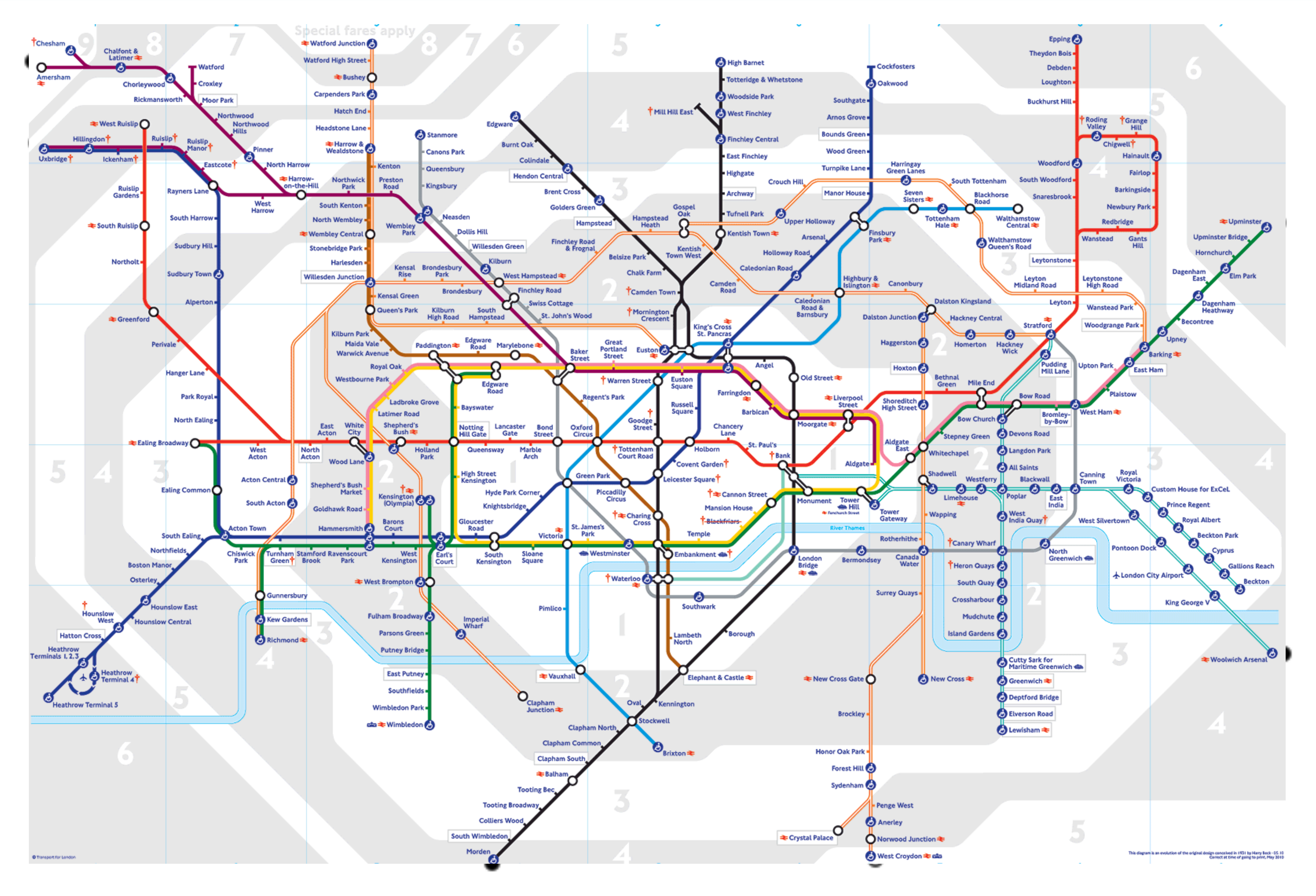
Why study graphs and graph algorithms?

- Hundreds of graph algorithms.
- Thousands of real-world applications.
- Fascinating branch of computer science and discrete math.



Transportation networks

Vertex = subway stop; edge = direct route.



London Underground (Tube) Map

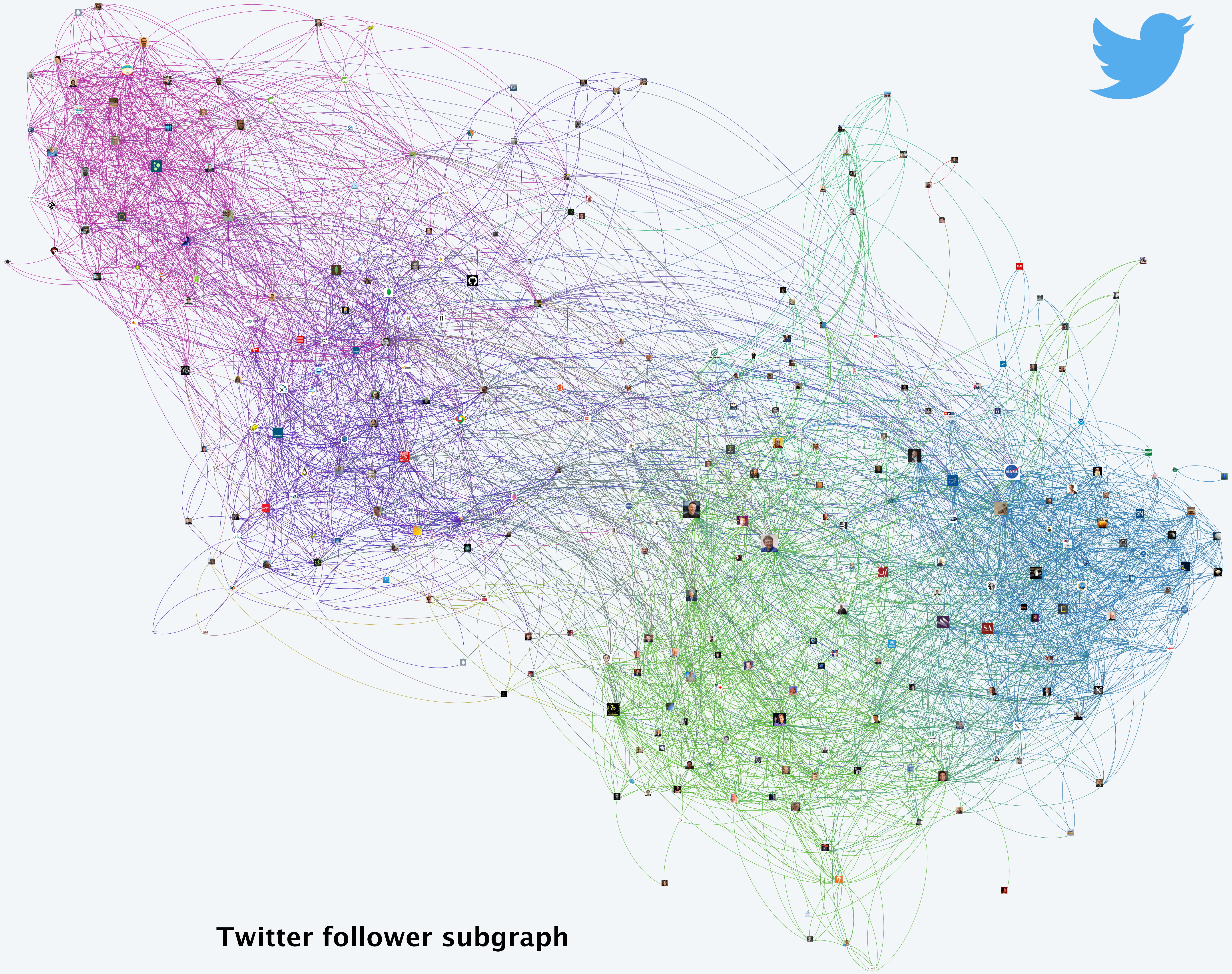
Social networks

Vertex = person; edge = social relationship.

"Visualizing Friendships" by Paul Butler

Twitter followers

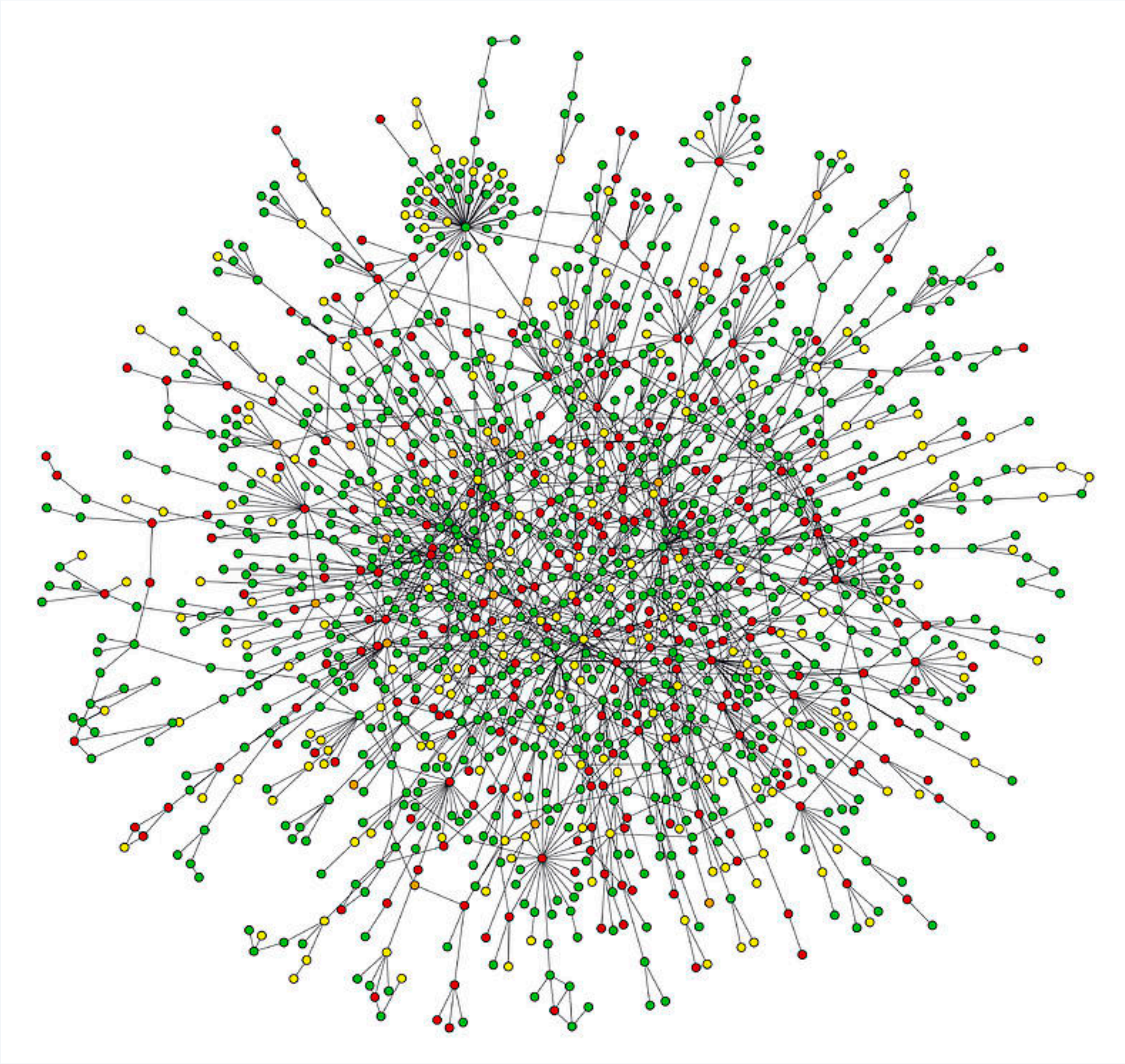
Vertex = Twitter account; edge = Twitter follower.



Twitter follower subgraph

Protein-protein interaction network

Vertex = protein; edge = interaction.



yeast protein interaction map

Graph applications

graph	vertex	edge
cell phone	phone	placed call
infectious disease	person	infection
financial	stock, currency	transactions
transportation	intersection	street
internet	router	fiber optic cable
web	web page	URL link
social relationship	person	friendship
object graph	object	pointer
protein network	protein	protein–protein interaction
circuit	logic gate	wire
neural network	neuron	synapse

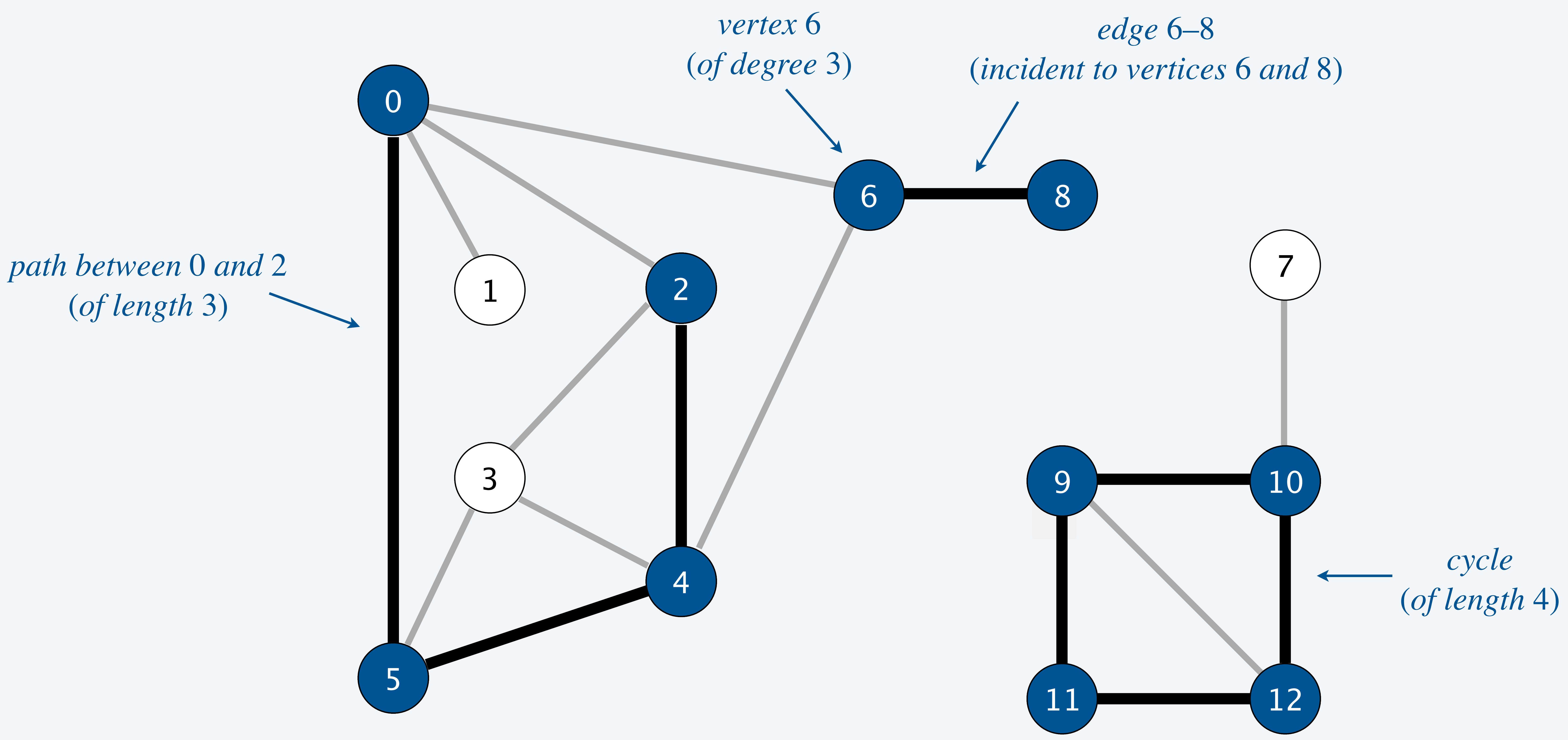
Undirected graph terminology

Graph. Set of vertices connected pairwise by edges.

Path. Sequence of vertices connected by edges, with no repeated edges.

Connected. Two vertices are connected if there is a path between them.

Cycle. Path (with ≥ 1 edge) whose first and last vertices are the same.



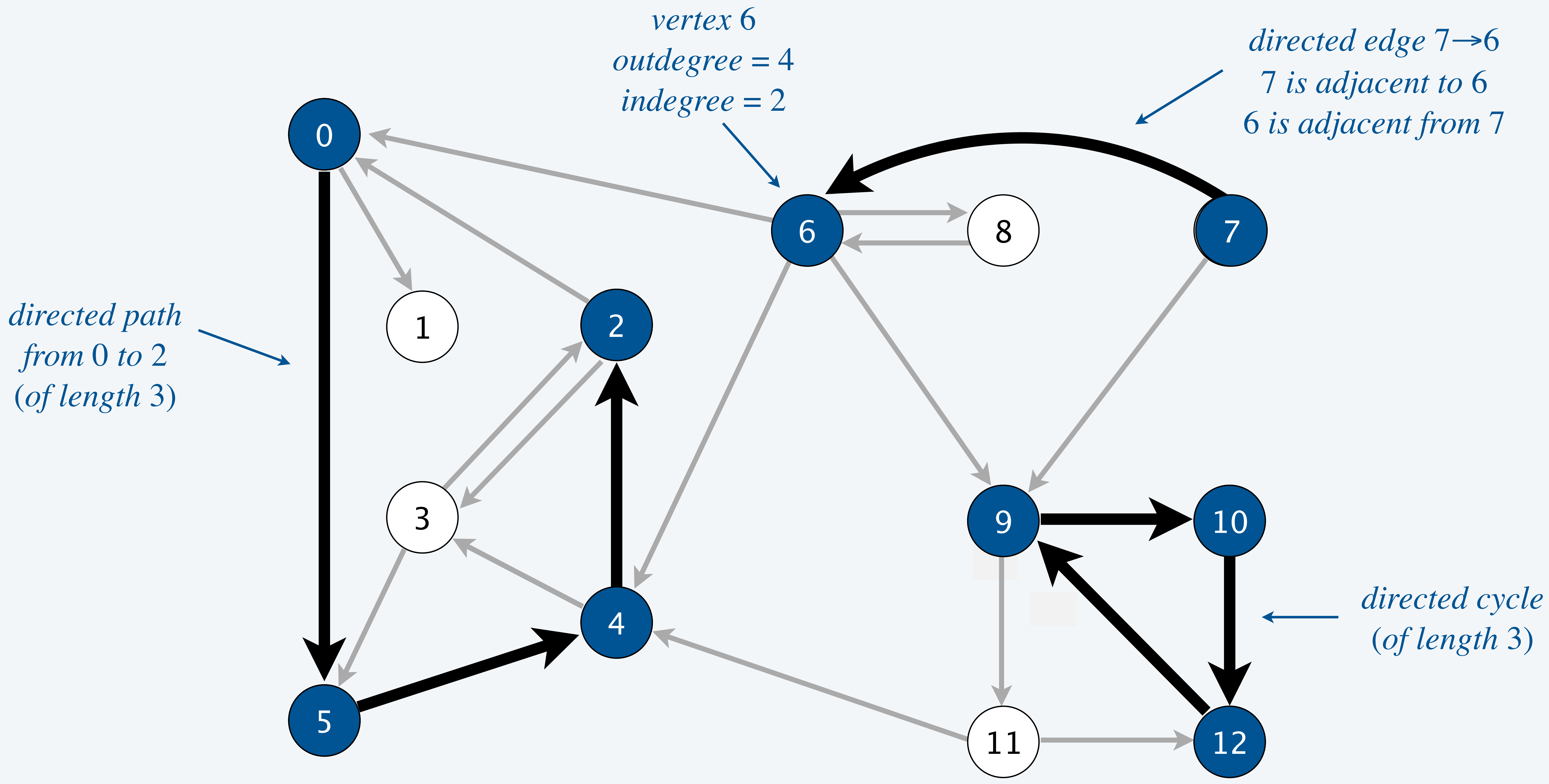
Directed graph terminology

Digraph. Set of vertices connected pairwise by directed edges.

Directed path. Sequence of vertices connected by directed edges, with no repeated edges.

Reachable. Vertex w is reachable from vertex v if there is a directed path from v to w.

Directed cycle. Directed path (with ≥ 1 edge) whose first and last vertices are the same.



Which of these graphs is best modeled as a directed graph?

A. Facebook: vertex = person; edge = friendship.

B. Web: vertex = webpage; edge = URL link.

C. Internet: vertex = router; edge = fiber optic cable.

D. Molecule: vertex = atom; edge = chemical bond.

Some graph-processing problems

	graph problem	description
😁	s–t path	*Find a path between s and t.*
😁	shortest s–t path	*Find a path with the fewest edges between s to t.*
😁	cycle	*Find a cycle.*
😁	Euler cycle	*Find a cycle that uses each edge exactly once.*
😈	Hamilton cycle	*Find a cycle that uses each vertex exactly once.*
😁	connected components	*Find connected components.*
🤷🏽‍♀️	graph isomorphism	*Find an isomorphism between two graphs.*
😀	planarity	*Draw in the plane with no crossing edges.*

also digraph versions

Challenge. Which problems are easy? Difficult? Intractable?

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4. Graphs and Digraphs I

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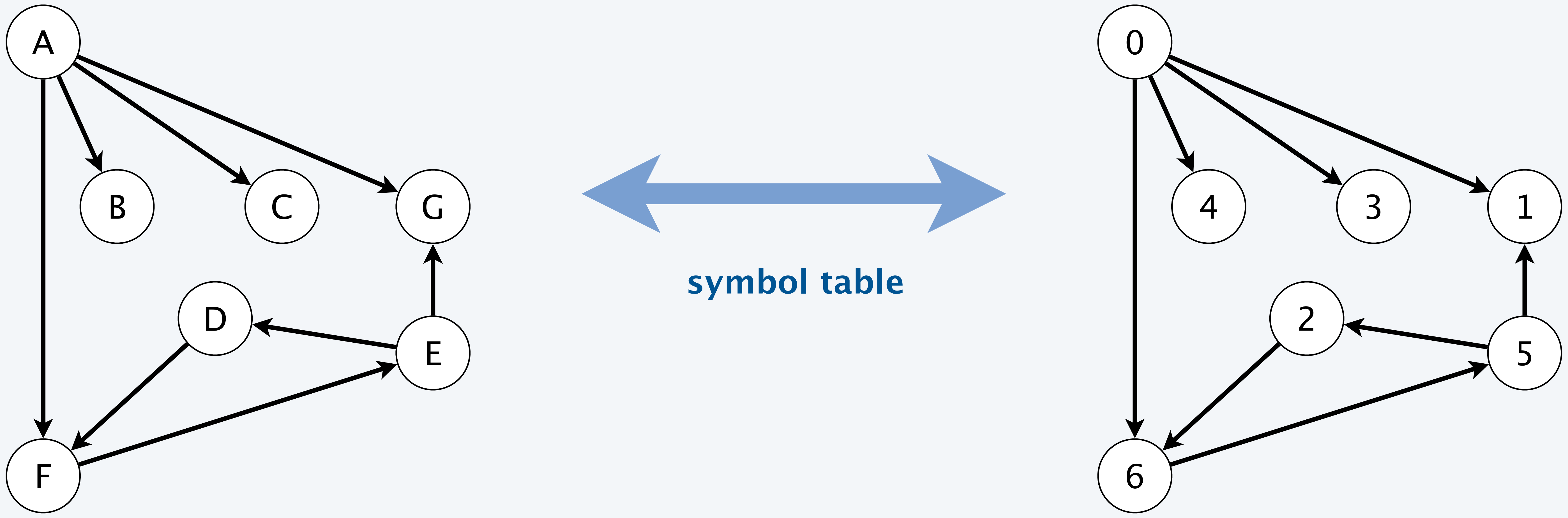
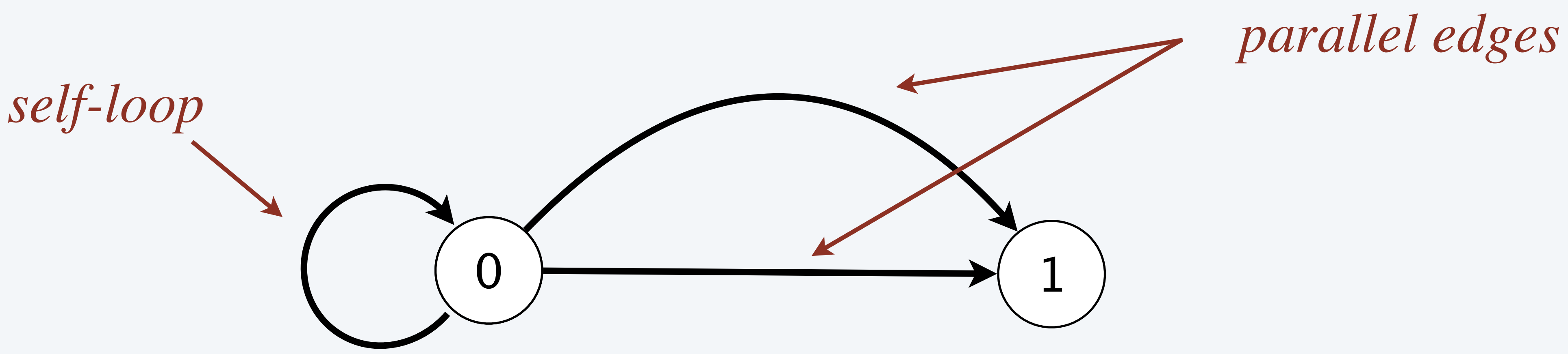
Digraph representation

Vertex representation.

- This lecture: integers between 0 and $V-1$.
- Real-world applications: use symbol table to convert between names and integers.



Def. A digraph is simple if it has no self-loops or parallel edges.



Digraph API

public class Digraph		
	`Digraph(int V)`	*create an empty digraph with V vertices*
`void`	`addEdge(int v, int w)`	*add a directed edge v→w* ← *our API allows self-loops and parallel edges*
`Iterable<Integer>`	`adj(int v)`	*vertices adjacent from v*
`int`	`V()`	*number of vertices*
`Digraph`	`reverse()`	*reverse digraph*
	⋮	⋮

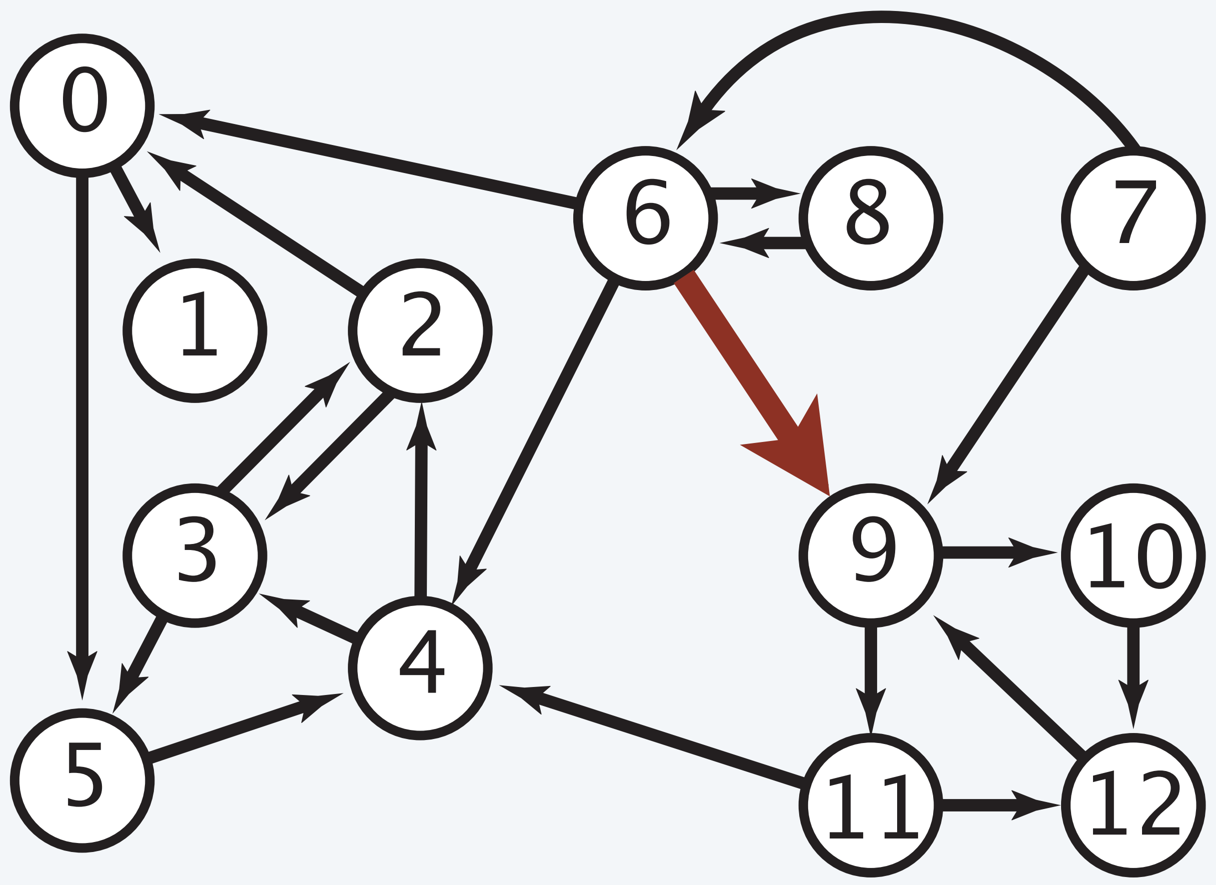
```
// outdegree of vertex v in digraph G
public static int outdegree(Digraph G, int v) {
    int count = 0;
    for (int w : G.adj(v))
       count++;
    return count;
}
```

Note: this method is in full `Digraph` *API, so no need to re-implement*

Digraph representation: adjacency matrix

Maintain a V-by-V boolean array; for each edge $v \to w$ in the digraph: `adj[v][w]` is `true`.

Memory. $\Theta(V^2)$ space.



`adj[][]`

to / **from**

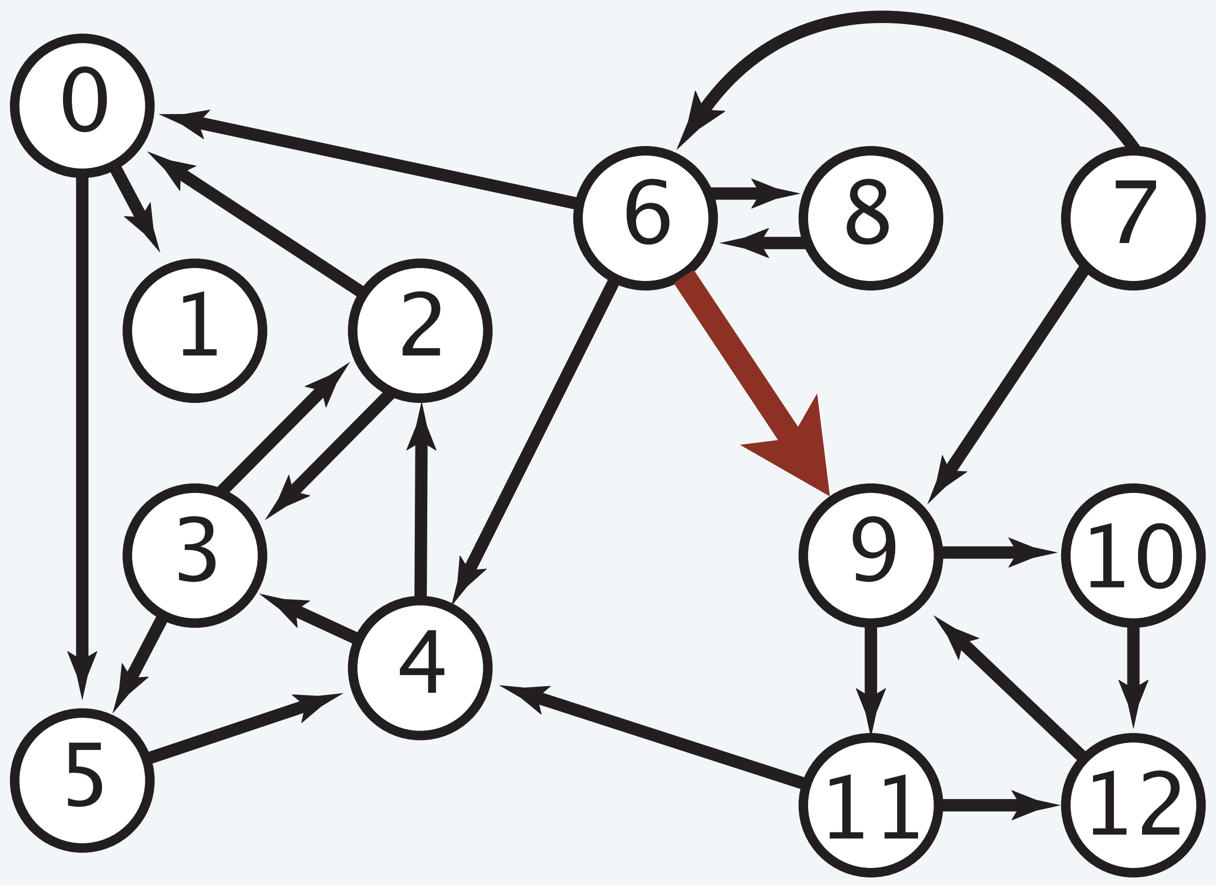
from \ to	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	1	0	0	0	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	1	1	0	0	0
7	0	0	0	0	0	0	1	0	0	1	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	0
10	0	0	0	0	0	0	0	0	0	0	0	0	1
11	0	0	0	0	1	0	0	0	0	0	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	0	0

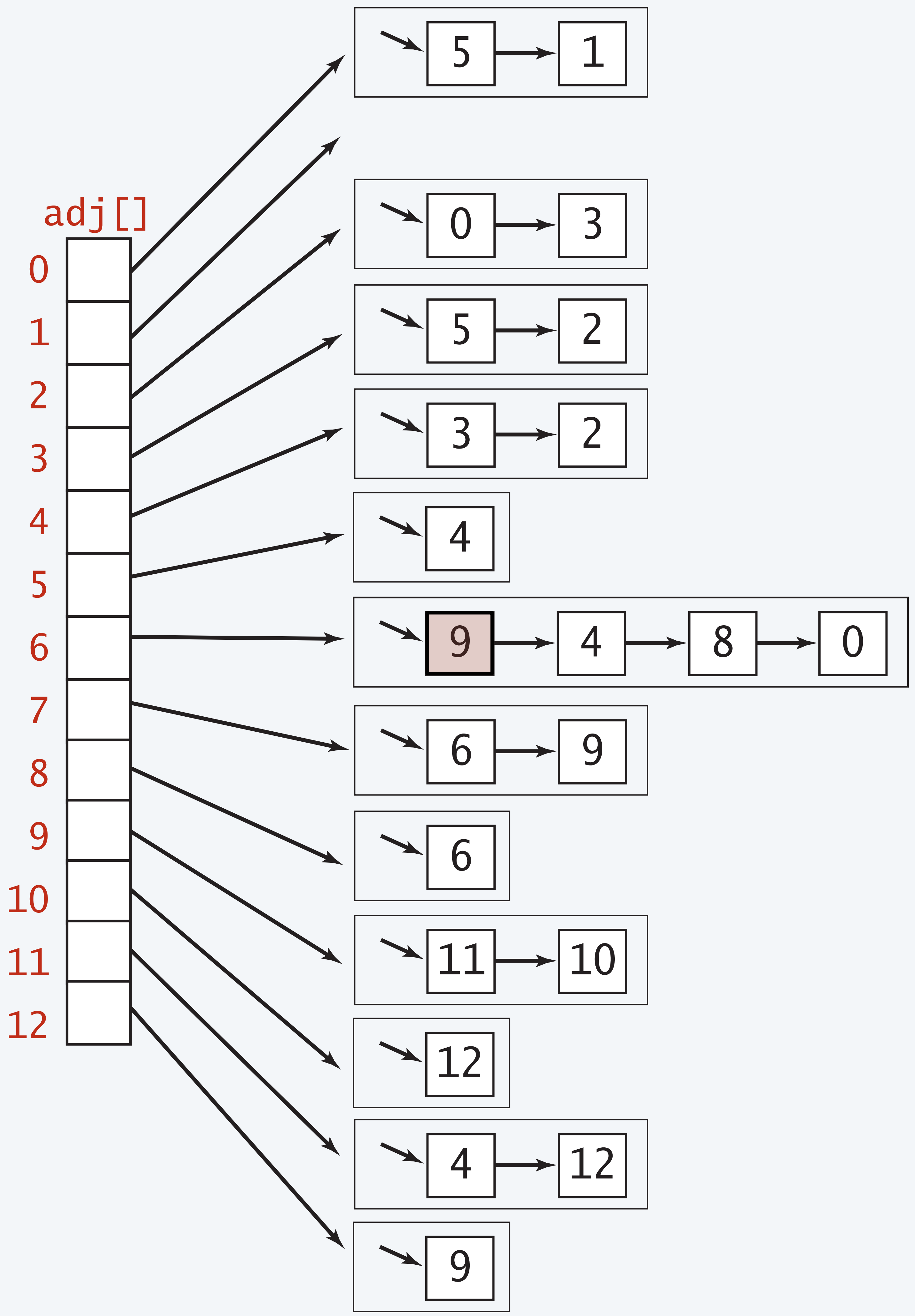
Note: parallel edges disallowed

Digraph representation: adjacency lists

Maintain vertex-indexed array of lists: `adj[v]` contains vertices adjacent from vertex `v`.

Memory. $\Theta(E + V)$ space.



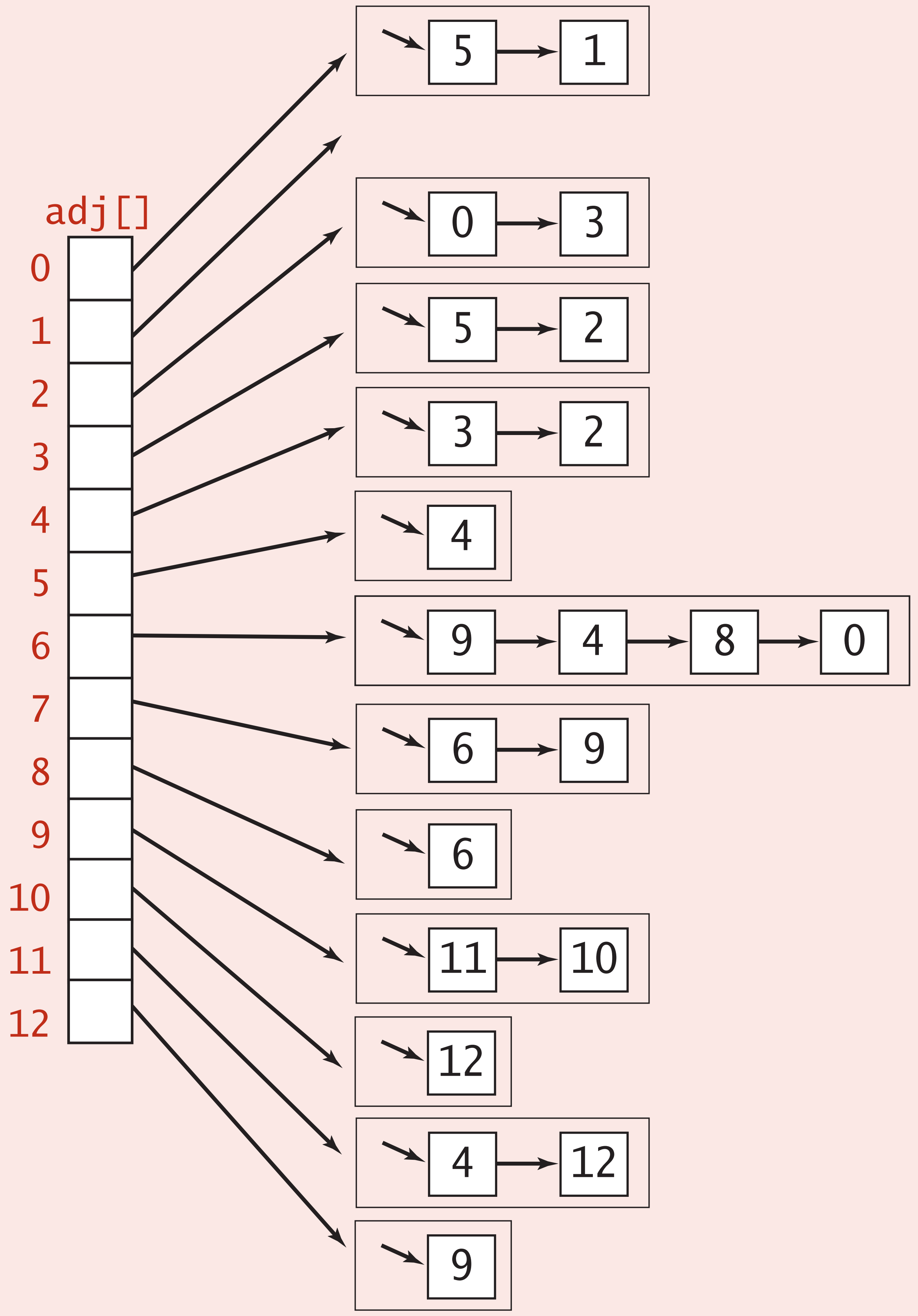


What is the running time of the following code fragment?

Assume adjacency-lists representation, V = # vertices, E = # edges.

```
for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "->" + w);
```

print each edge once



A. $\Theta(V)$

B. $\Theta(E + V)$

C. $\Theta(V^2)$

D. $\Theta(E\ V)$

Digraph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent from v.
- Real-world graphs tend to be sparse (not dense).

sparse: $\Theta(V)$ *edges*   dense: $\Theta(V^2)$ *edges*

representation	space	add edge from v to w	has edge from v to w?	iterate over vertices adjacent from v?
adjacency matrix	V^2	1	1	V †
adjacency lists	$E + V$	1	$outdegree(v)$	$outdegree(v)$

† *disallows parallel edges*

Digraph representation (adjacency lists): Java implementation

```
public class Digraph {

    private final int V;
    private Bag<Integer>[] adj;

    public Digraph(int V) {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<>();
    }

    public void addEdge(int v, int w) {
       adj[v].add(w);
    }

    public Iterable<Integer> adj(int v) {
       return adj[v];
    }

}
```

adjacency lists
(could use a stack or queue instead of a bag)

create empty digraph with V vertices

add edge v→w
(parallel edges and self-loops allowed)

iterator for vertices adjacent from v

https://algs4.cs.princeton.edu/42digraph/Digraph.java.html

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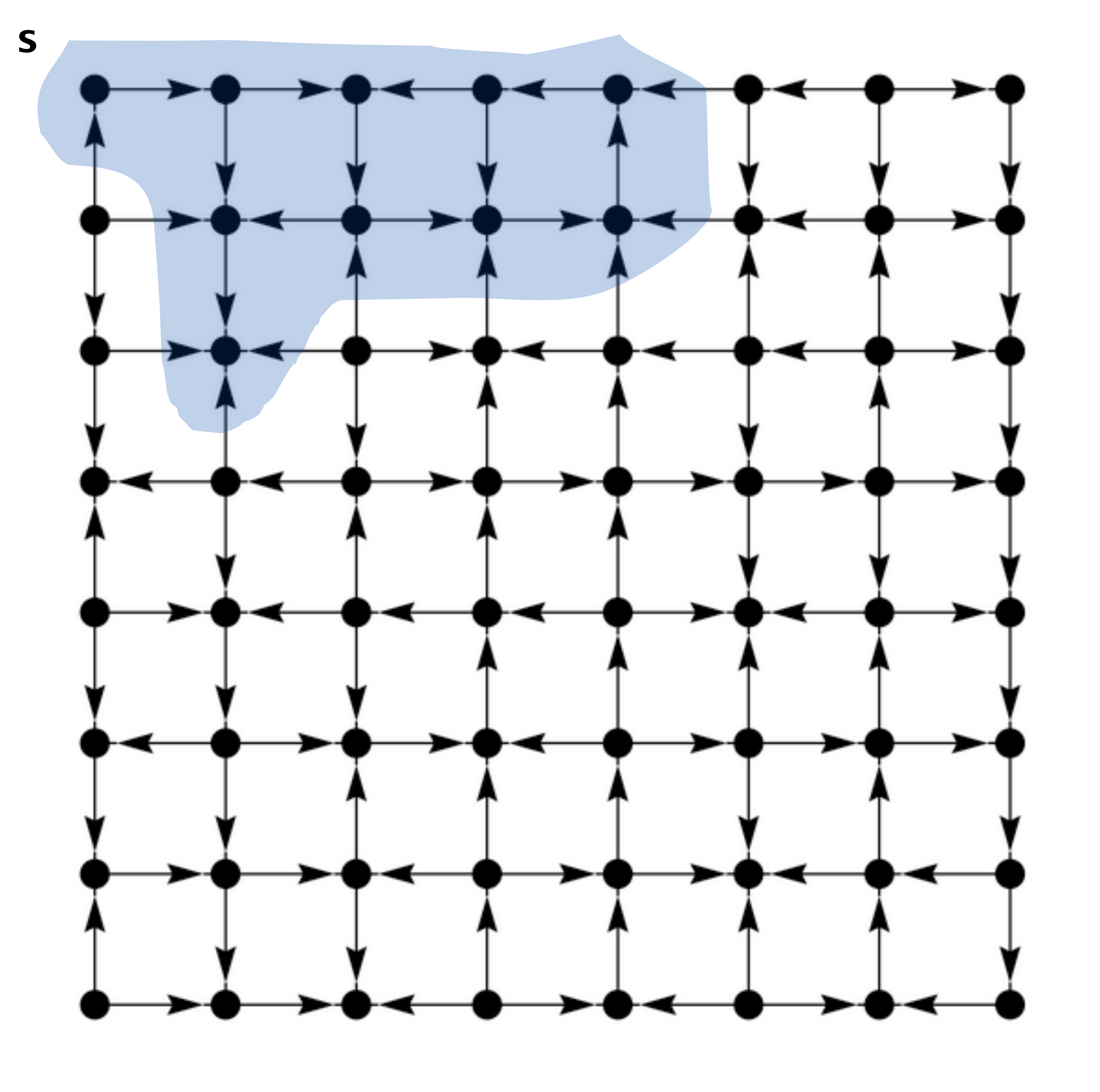
Reachability problem in a digraph

Reachability problem. Given a digraph G and vertex s, find all vertices reachable from s.



Reachability problem in a digraph

Reachability problem. Given a digraph G and vertex s, find all vertices reachable from s.

Depth-first search. A systematic method to explore all vertices reachable from s.

DFS (to visit a vertex v)

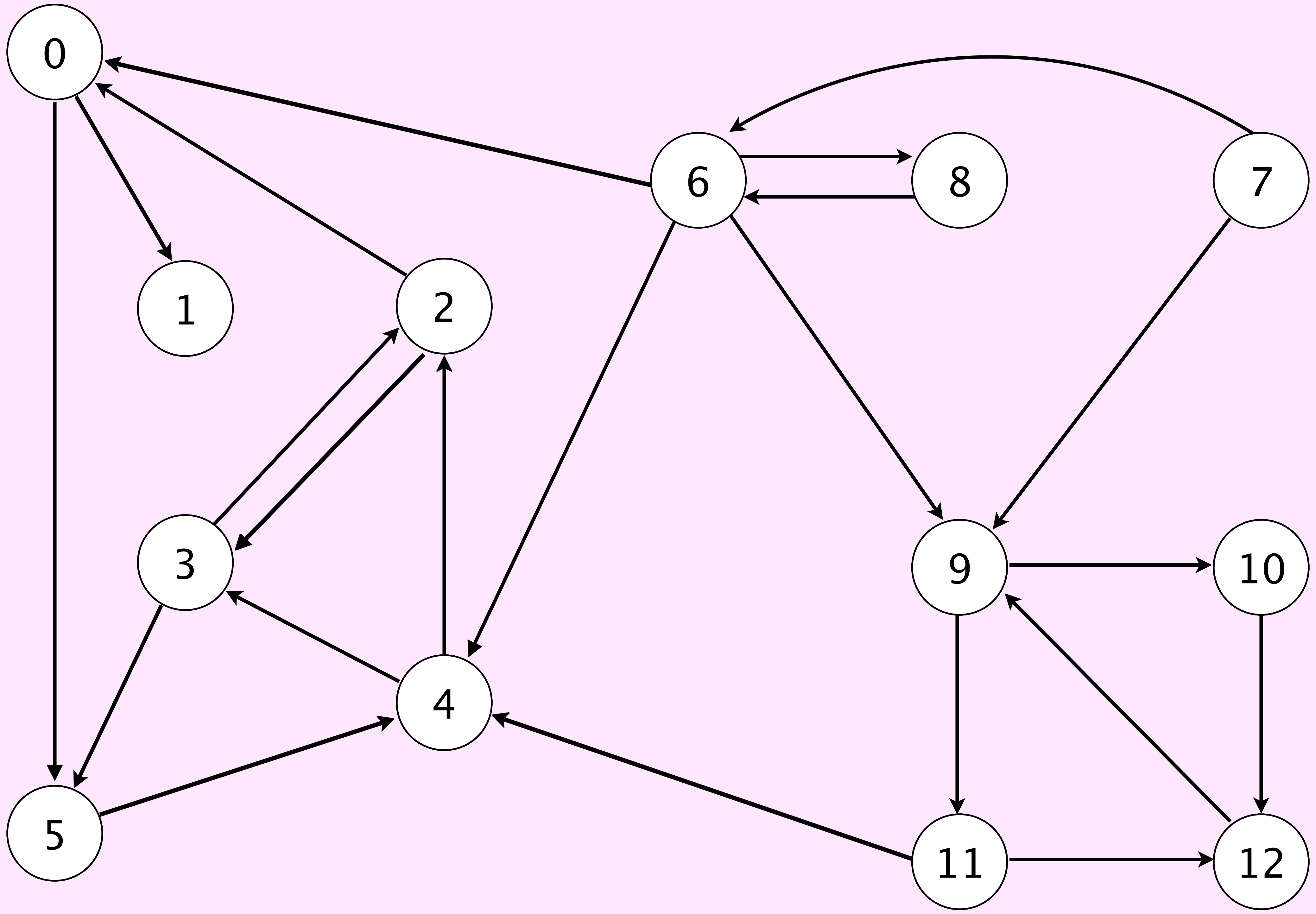
Mark vertex v.

Recursively visit all unmarked vertices w adjacent from v.

Directed depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent from v.



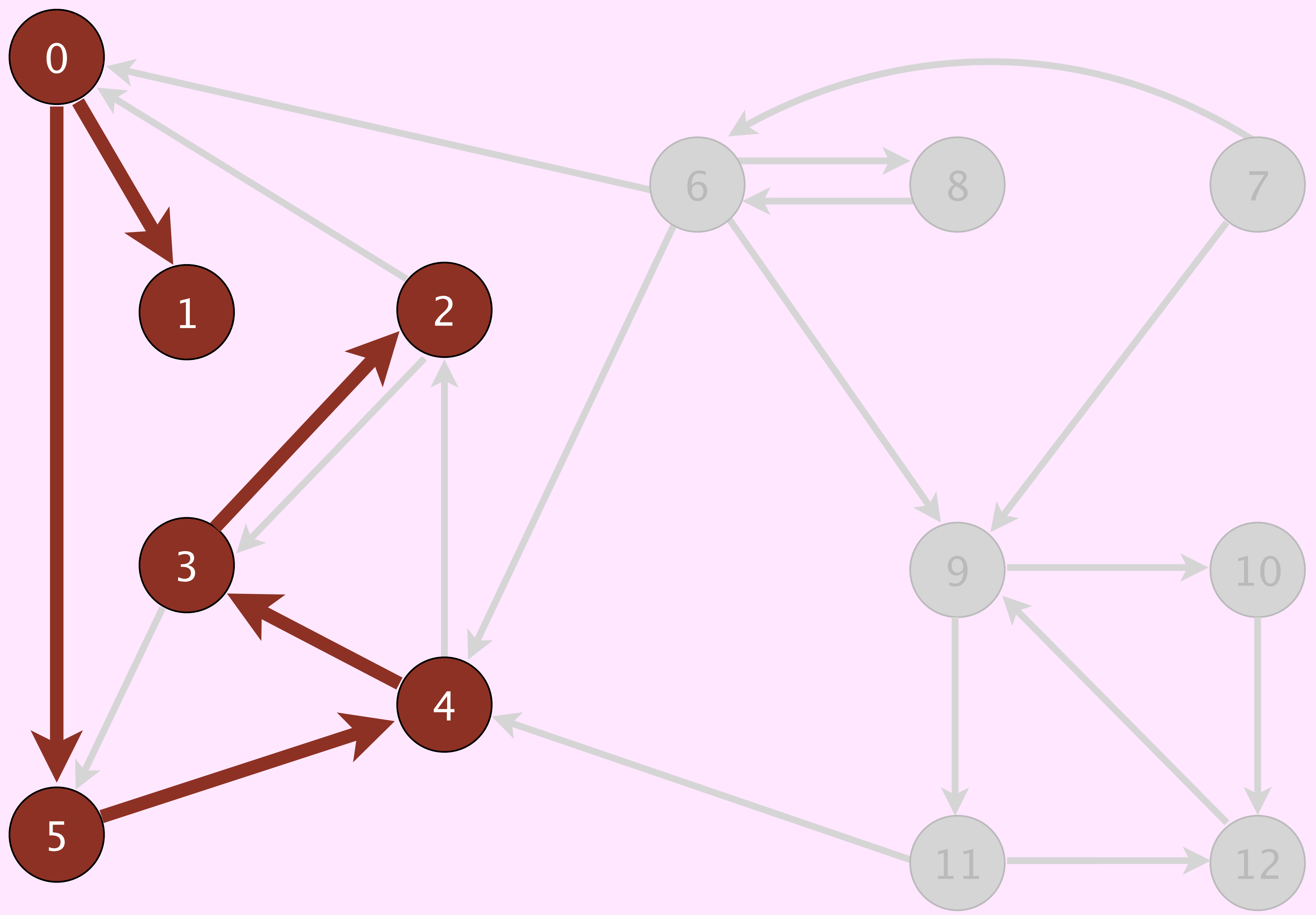
a directed graph

4→2
2→3
3→2
6→0
0→1
2→0
11→12
12→9
9→10
9→11
8→9
10→12
11→4
4→3
3→5
6→8
8→6
5→4
0→5
6→4
6→9
7→6

Directed depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent from v.



reachable from 0

v	marked[]
0	T
1	T
2	T
3	T
4	T
5	T
6	F
7	F
8	F
9	F
10	F
11	F
12	F

reachable from vertex 0

Run DFS using the given adjacency-lists representation of digraph G, starting at vertex 0. In which order is dfs(G, v) called?

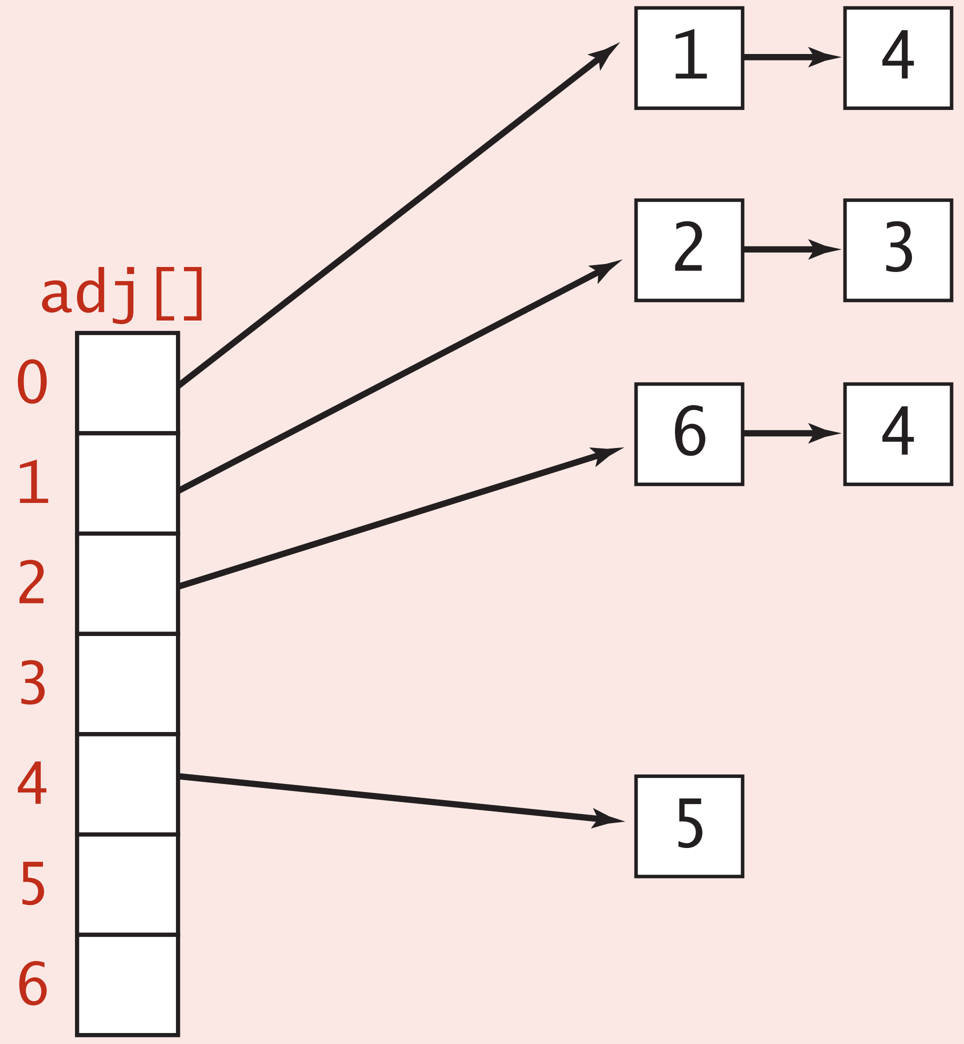
DFS preorder

A. 0 1 2 4 5 3 6

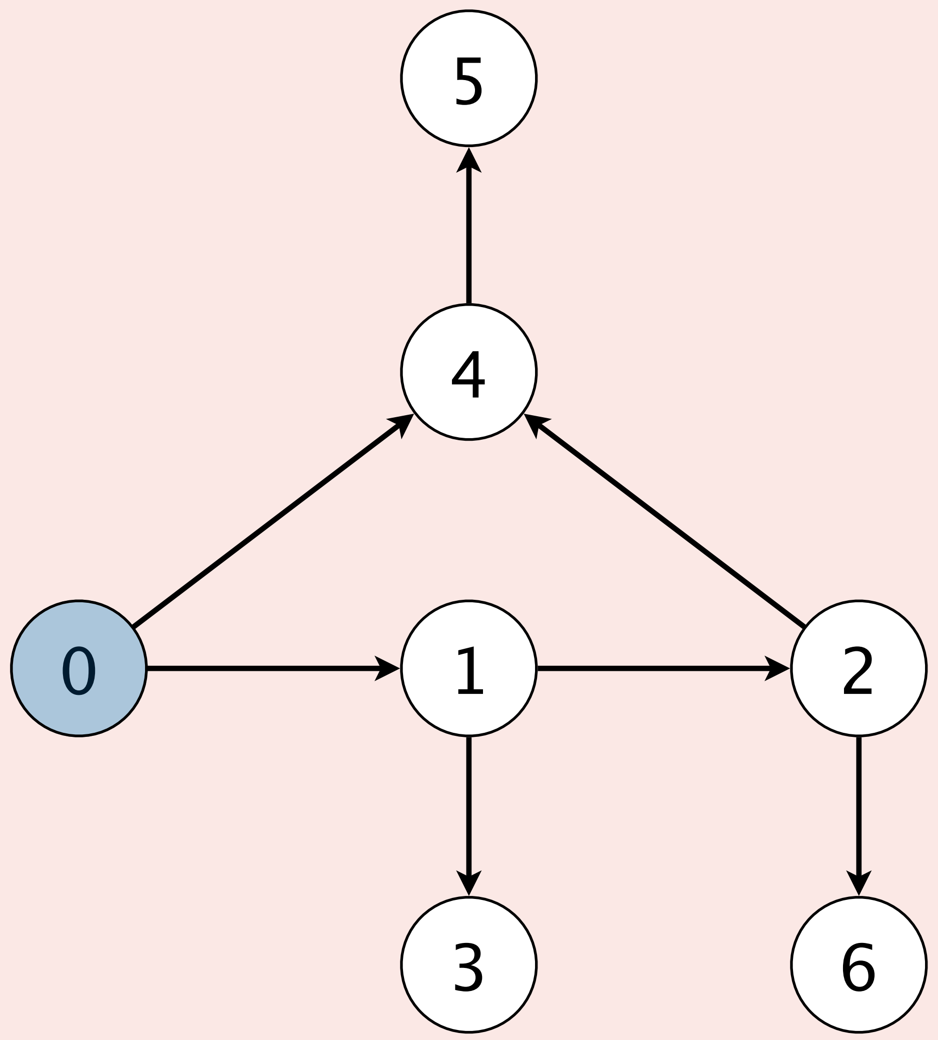
B. 0 1 2 4 5 6 3

C. 0 1 3 2 6 4 5

D. 0 1 2 6 4 5 3



adjacency-lists representation



digraph G

Depth-first search: Java implementation

```
public class DirectedDFS {

    private boolean[] marked;

    public DirectedDFS(Digraph G, int s) {
        marked = new boolean[G.V()];
        dfs(G, s);
    }

    private void dfs(Digraph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w])
                dfs(G, w);
    }

    public boolean isReachable(int v) {
        return marked[v];
    }

}
```

`marked[v]` = `true` *if v is reachable from s*

constructor marks vertices reachable from s

recursive DFS does the work

is v reachable from s ?

https://algs4.cs.princeton.edu/42digraph/DirectedDFS.java.html

Depth-first search: running time

Proposition. DFS marks all vertices reachable from s in $\Theta(E + V)$ time in the worst case.

Pf.

- Initializing the `marked[]` array takes $\Theta(V)$ time.
- Each vertex is visited at most once.
- Visiting a vertex takes time proportional to its outdegree:

$$outdegree(v_0) + outdegree(v_1) + outdegree(v_2) + \ldots = E$$

in worst case,
all V vertices are reachable from s

Note. If all vertices are reachable from s, then $E \geq V - 1$ and running time simplifies to $\Theta(E)$.

What could happen if we marked a vertex at the end of the DFS call (instead of beginning)?

A. Marks a vertex not reachable from s.

B. Compile-time error.

C. Infinite loop / stack overflow.

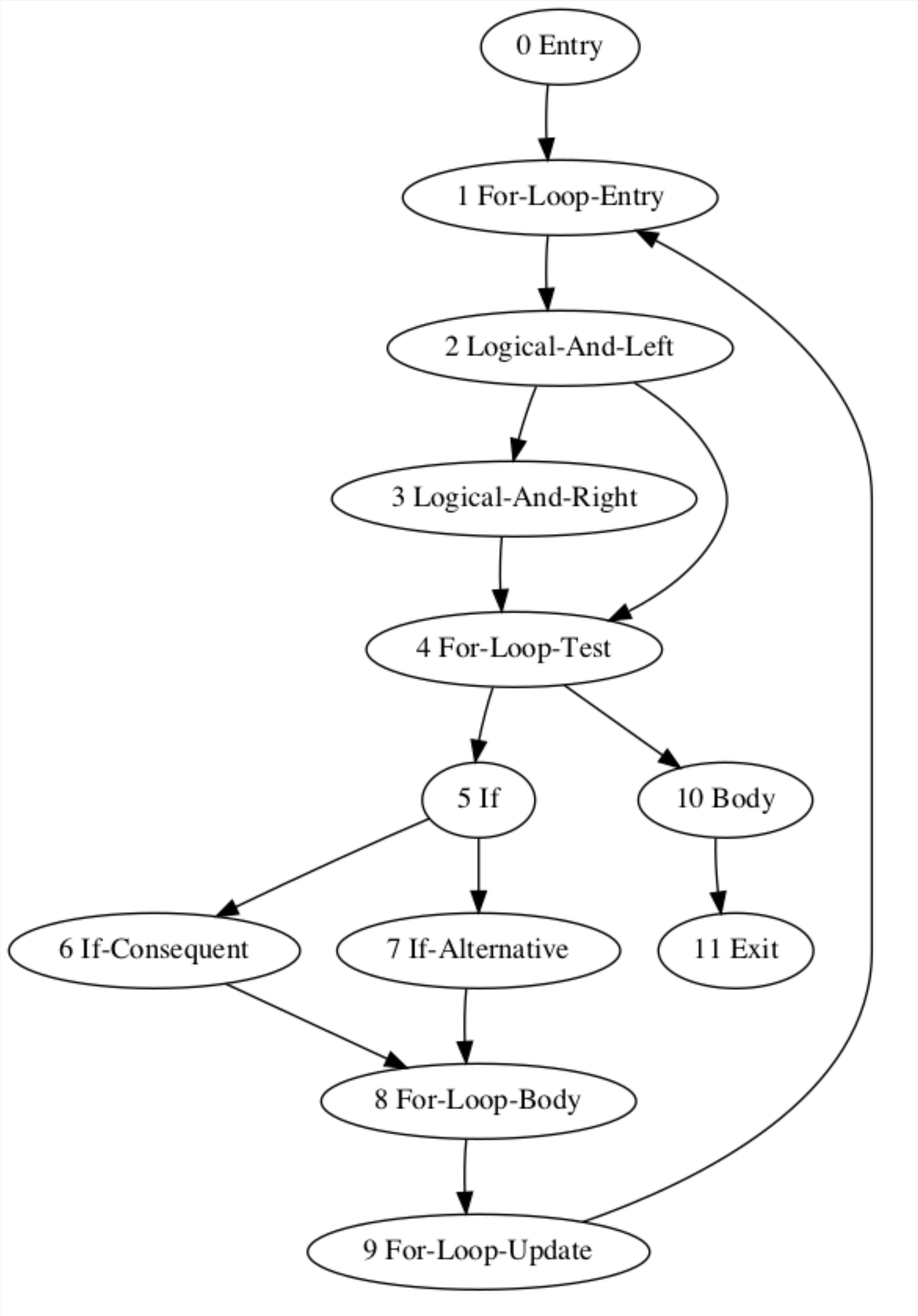
D. None of the above.

```
private void dfs(Digraph G, int v) {
  marked[v] = true;
  for (int w : G.adj(v))
     if (!marked[w])
        dfs(G, w);
  marked[v] = true;
}
```

Reachability application: program control-flow analysis

Every program is a digraph.

- Vertex = basic block of instructions (straight-line program).
- Edge = jump.



Dead-code elimination.

Find (and remove) unreachable code.

Infinite-loop detection.

Determine whether exit is unreachable.

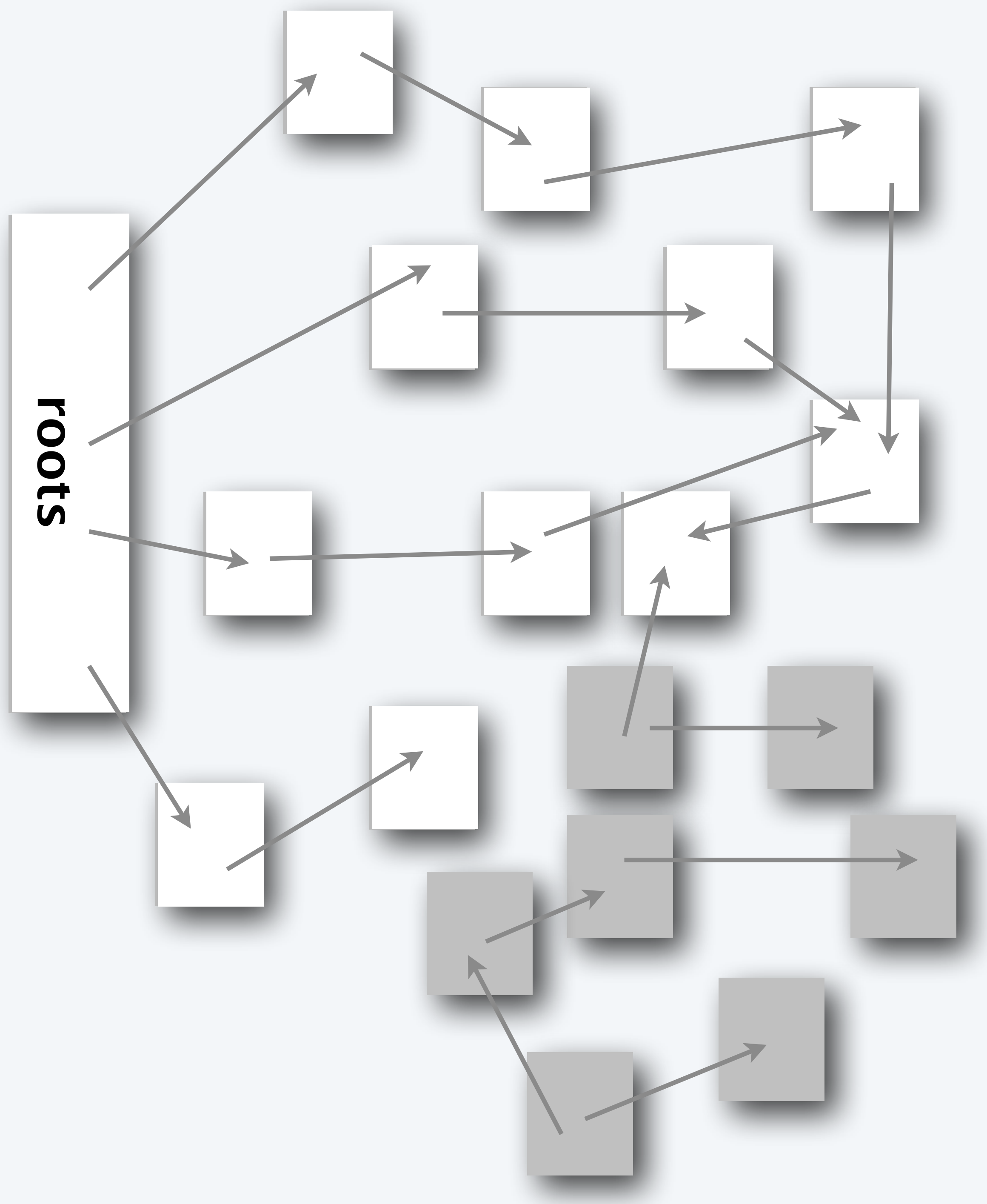
Reachability application: mark-sweep garbage collector

Every data structure is a digraph.

- Vertex = object.
- Edge = reference/pointer.

Roots. Objects known to be directly accessible by program (e.g., stack frame).

Reachable objects. Objects indirectly accessible by program (starting at a root and following a chain of pointers).

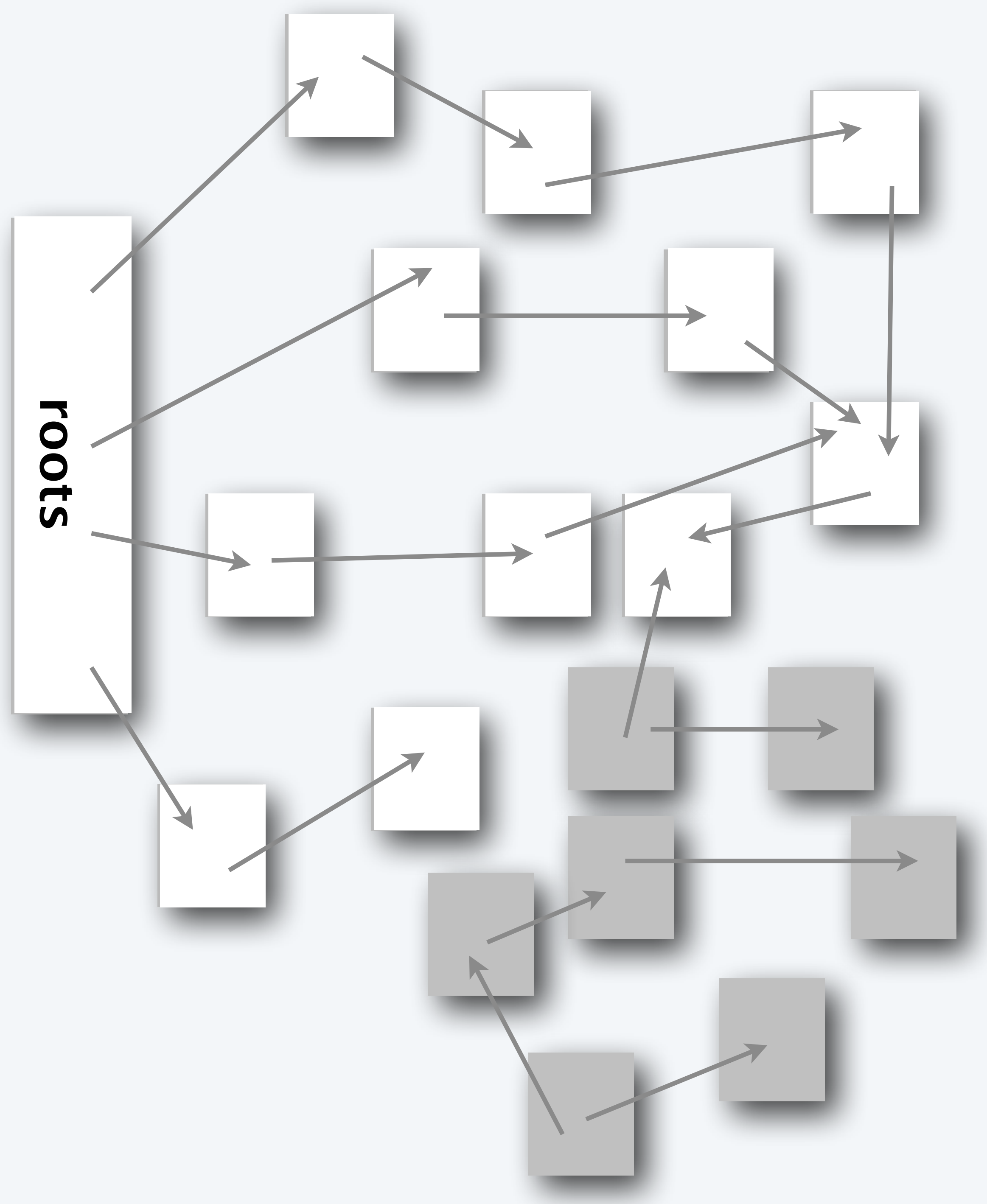


Reachability application: mark-sweep garbage collector

Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS function-call stack).



Algorithms

Robert Sedgewick | Kevin Wayne

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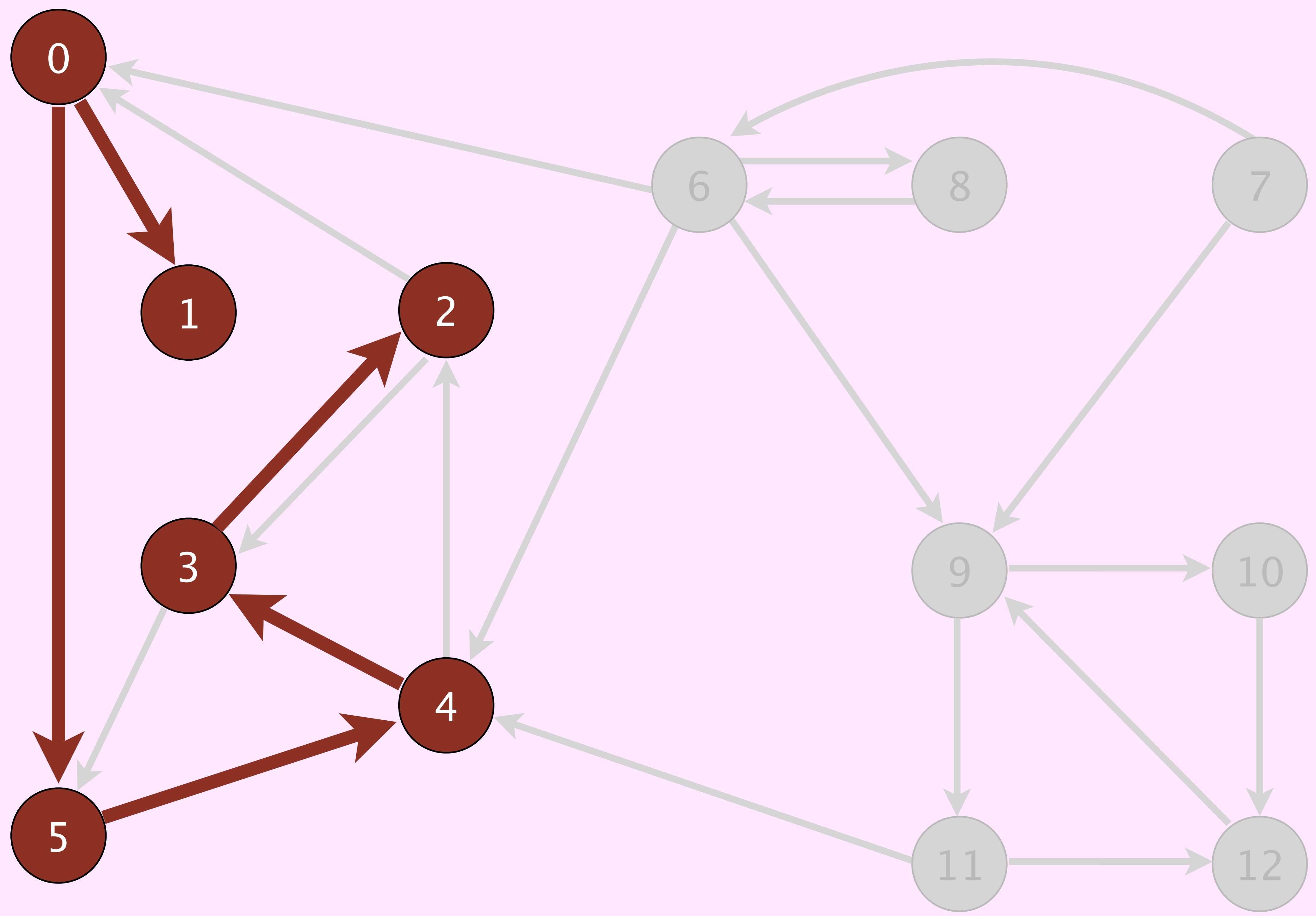
Directed paths DFS demo

Goal. DFS determines which vertices are reachable from s. How to reconstruct paths?

Solution. Use parent-link representation.



reachable from 0

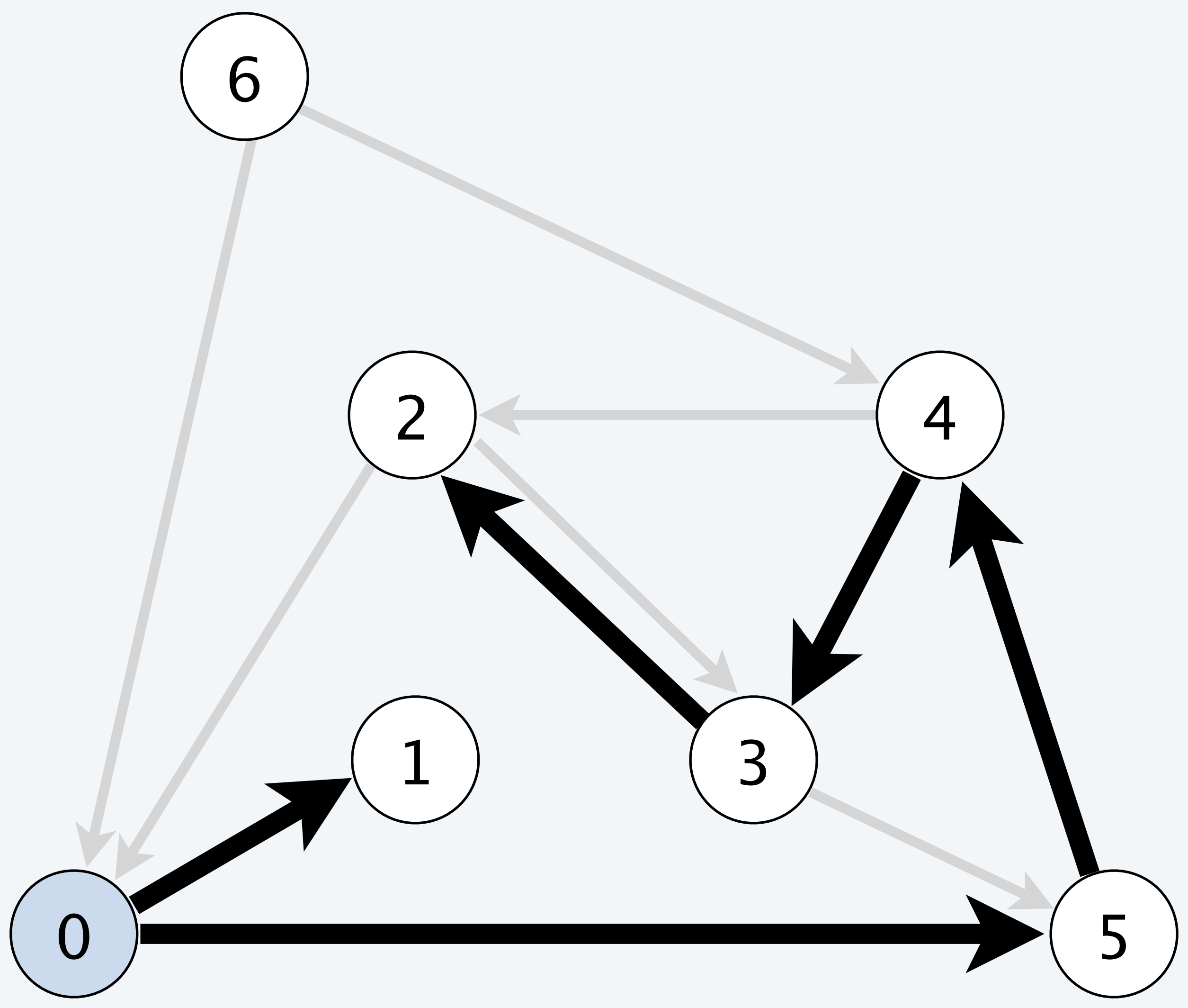
v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

parent-link representation of paths from vertex 0

Depth-first search: path finding

Parent-link representation of paths from s.

- Maintain an integer array `edgeTo[]`.
- Interpretation: `edgeTo[v]` is the next-to-last vertex on a path from s to v.
- To reconstruct path from s to v, trace `edgeTo[]` backward from v to s (and reverse).



v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–

```
public Iterable<Integer> pathTo(int v) {
   if (!marked[v]) return null;
   Stack<Integer> path = new Stack<>();
   for (int x = v; x != s; x = edgeTo[x])
      path.push(x);
   path.push(s);
   return path;
}
```

Depth-first search (with path finding): Java implementation

```
public class DepthFirstDirectedPaths {

    private boolean[] marked;
    private int[] edgeTo;
    private int s;

    public DepthFirstDirectedPaths(Digraph G, int s) {
        ...
        dfs(G, s);
    }

    private void dfs(Digraph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v)) {
            if (!marked[w]) {
                dfs(G, w);
                edgeTo[w] = v;
            }
        }
    }
}
```

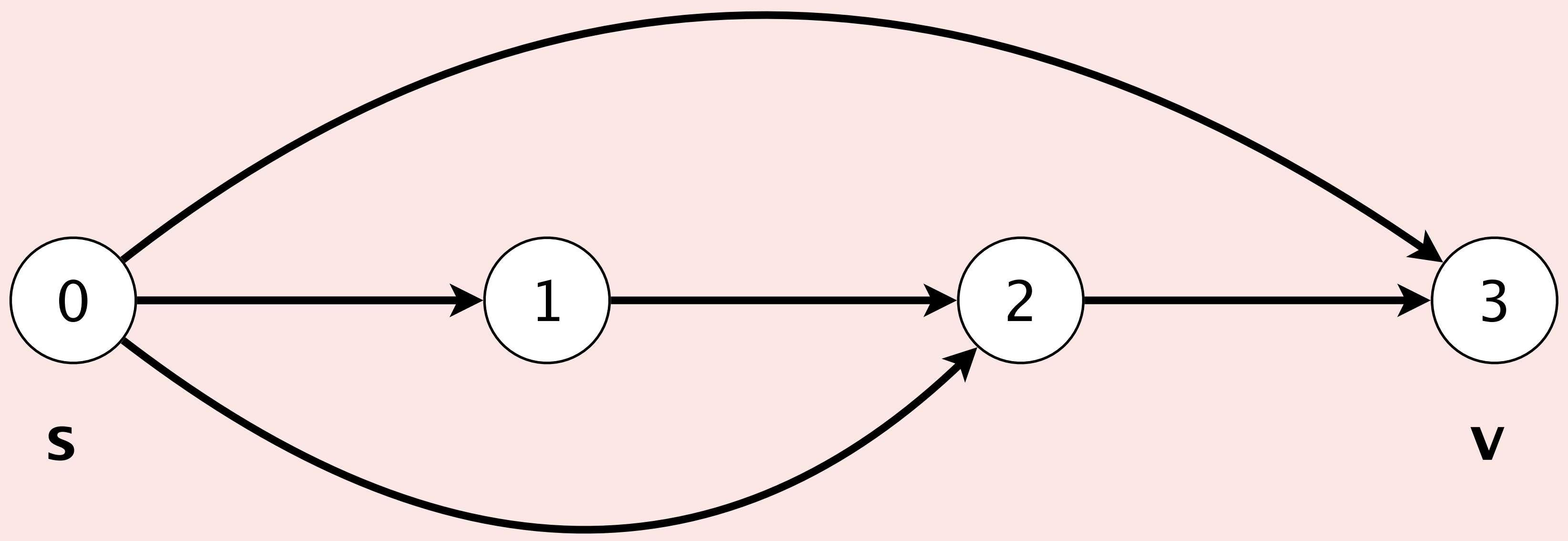
edgeTo[v] = *previous vertex on path from s to v*

v→w is edge that led to the discovery of w

https://algs4.cs.princeton.edu/42digraph/DepthFirstDirectedPaths.java.html

Suppose there are many paths from s to v. Which one does DepthFirstDirectedPaths find?

- **A.** A shortest path (fewest edges).
- **B.** A longest path (most edges).
- **C.** Depends on digraph representation.



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4. Graphs and Digraphs I

Flood fill

Problem. Implement flood fill (Photoshop magic wand).

Depth-first search in undirected graphs

Connectivity problem. Given an undirected graph G and vertex s, find all vertices connected to s.

Solution. Use DFS. ← *but now, for each undirected edge v–w: v is adjacent to w and w is adjacent to v*

DFS (to visit a vertex v)

Mark vertex v.

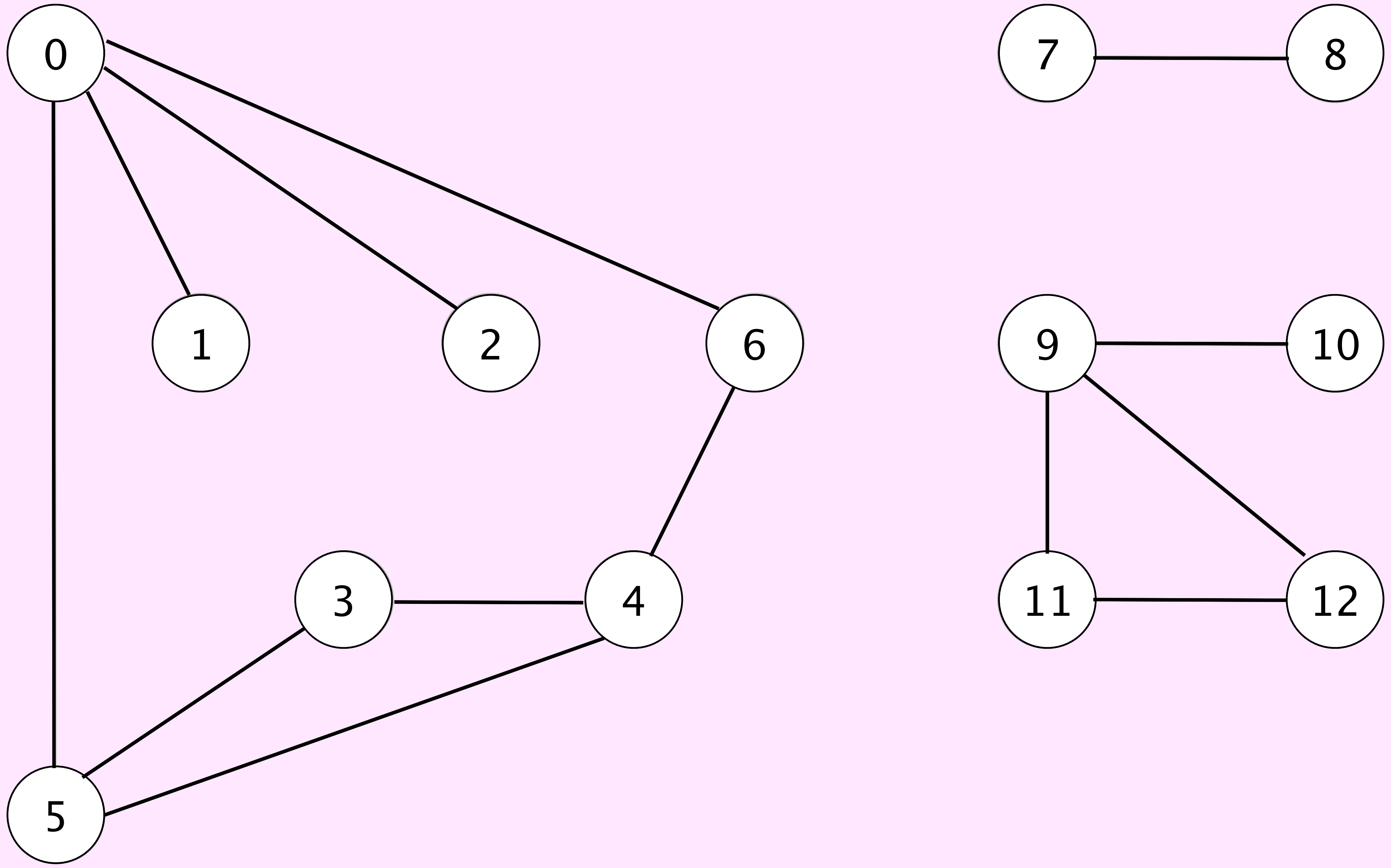
Recursively visit all unmarked vertices w adjacent to v.

Proposition. DFS marks all vertices connected to s in $\Theta(E + V)$ time in the worst case.

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



graph G

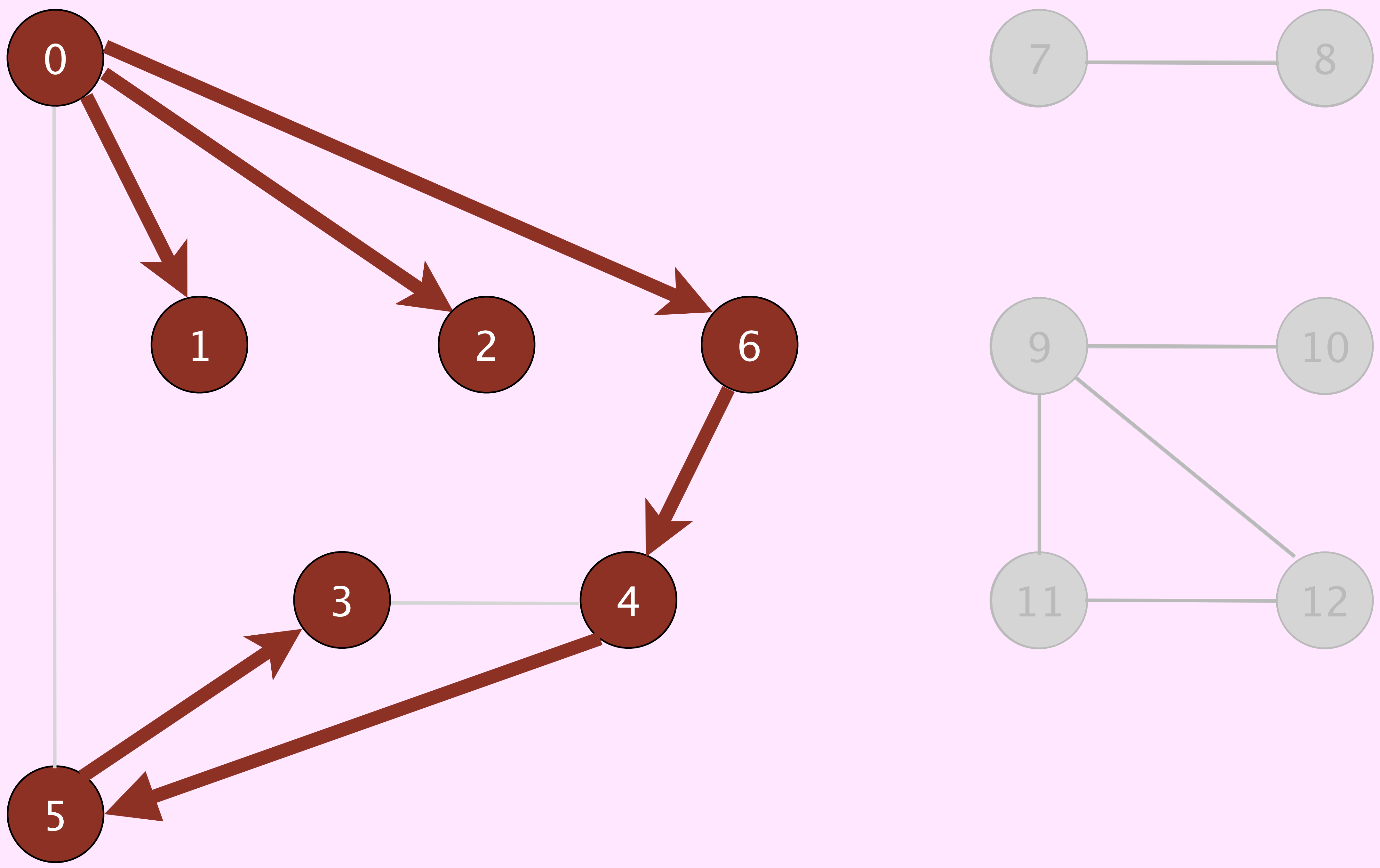
tinyG.txt

```
V → 13
    13 ← E
    0 5
    4 3
    0 1
    9 12
    6 4
    5 4
    0 2
    11 12
    9 10
    0 6
    7 8
    9 11
    5 3
```

Depth-first search demo

To visit a vertex v :

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.



vertices connected to 0
(and associated paths)

v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

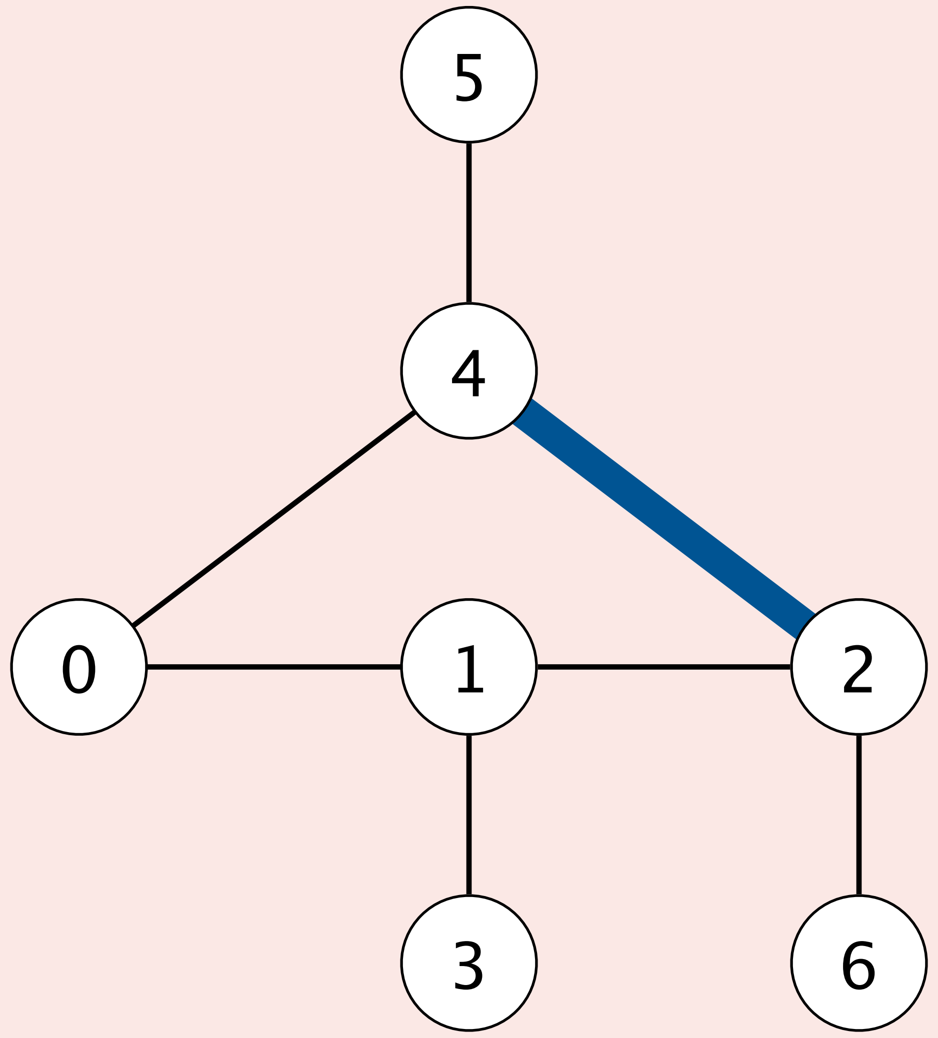
How to represent an undirected edge v–w using adjacency lists?

A. Add w to adjacency list for v.

B. Add v to adjacency list for w.

C. Both A and B.

D. None of the above.



Directed graph representation (review)

```
public class Digraph {

    private final int V;
    private Bag<Integer>[] adj;

    public Digraph(int V) {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<>();
    }

    public void addEdge(int v, int w) {
       adj[v].add(w);

    }

    public Iterable<Integer> adj(int v) {
       return adj[v];
    }
}
```

adjacency lists

create empty digraph with V vertices

add edge v→w

iterator for vertices adjacent from v

https://algs4.cs.princeton.edu/42digraph/Digraph.java.html

Undirected graph representation

```
public class Graph {

    private final int V;
    private Bag<Integer>[] adj;

    public Graph(int V) {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<>();
    }

    public void addEdge(int v, int w) {
       adj[v].add(w);
       adj[w].add(v);
    }

    public Iterable<Integer> adj(int v) {
       return adj[v];
    }
}
```

adjacency lists

create empty graph with V vertices

add edge v–w

iterator for vertices adjacent to v

https://algs4.cs.princeton.edu/41graph/Graph.java.html

Depth-first search (in directed graphs)

```
public class DirectedDFS {

    private boolean[] marked;

    public DirectedDFS(Digraph G, int s) {
      marked = new boolean[G.V()];
      dfs(G, s);
    }

    private void dfs(Digraph G, int v) {
       marked[v] = true;
       for (int w : G.adj(v))
          if (!marked[w])
             dfs(G, w);
    }

    public boolean isReachable(int v) {
       return marked[v];
    }

}
```

`marked[v]` = `true` *if v is reachable from s*

constructor marks vertices reachable from s

recursive DFS does the work

is v reachable from s ?

https://algs4.cs.princeton.edu/42digraph/DirectedDFS.java.html

Depth-first search (in undirected graphs)

```
public class DepthFirstSearch {

    private boolean[] marked;

    public DirectedDFS(Graph G, int s) {
      marked = new boolean[G.V()];
      dfs(G, s);
    }

    private void dfs(Graph G, int v) {
       marked[v] = true;
       for (int w : G.adj(v))
          if (!marked[w])
             dfs(G, w);
    }

    public boolean isConnected(int v) {
       return marked[v];
    }

}
```

`marked[v]` = `true` *if v is connected to s*

constructor marks vertices connected to s

recursive DFS does the work

is v connected to s ?

https://algs4.cs.princeton.edu/41graph/DepthFirstSearch.java.html

Depth-first search summary

DFS enables direct solution of several elementary graph and digraph problems.

- Reachability (in a digraph). ✓
- Connectivity (in a graph). ✓
- Path finding (in a graph or digraph). ✓
- Topological sort. ← *next lecture*
- Directed cycle detection. ← *precept*

DFS is also core of solution to more advanced problems.

- Euler cycle.
- Biconnectivity.
- 2-satisfiability.
- Planarity testing.
- Strong components.
- Nonbipartite matching.
- …

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

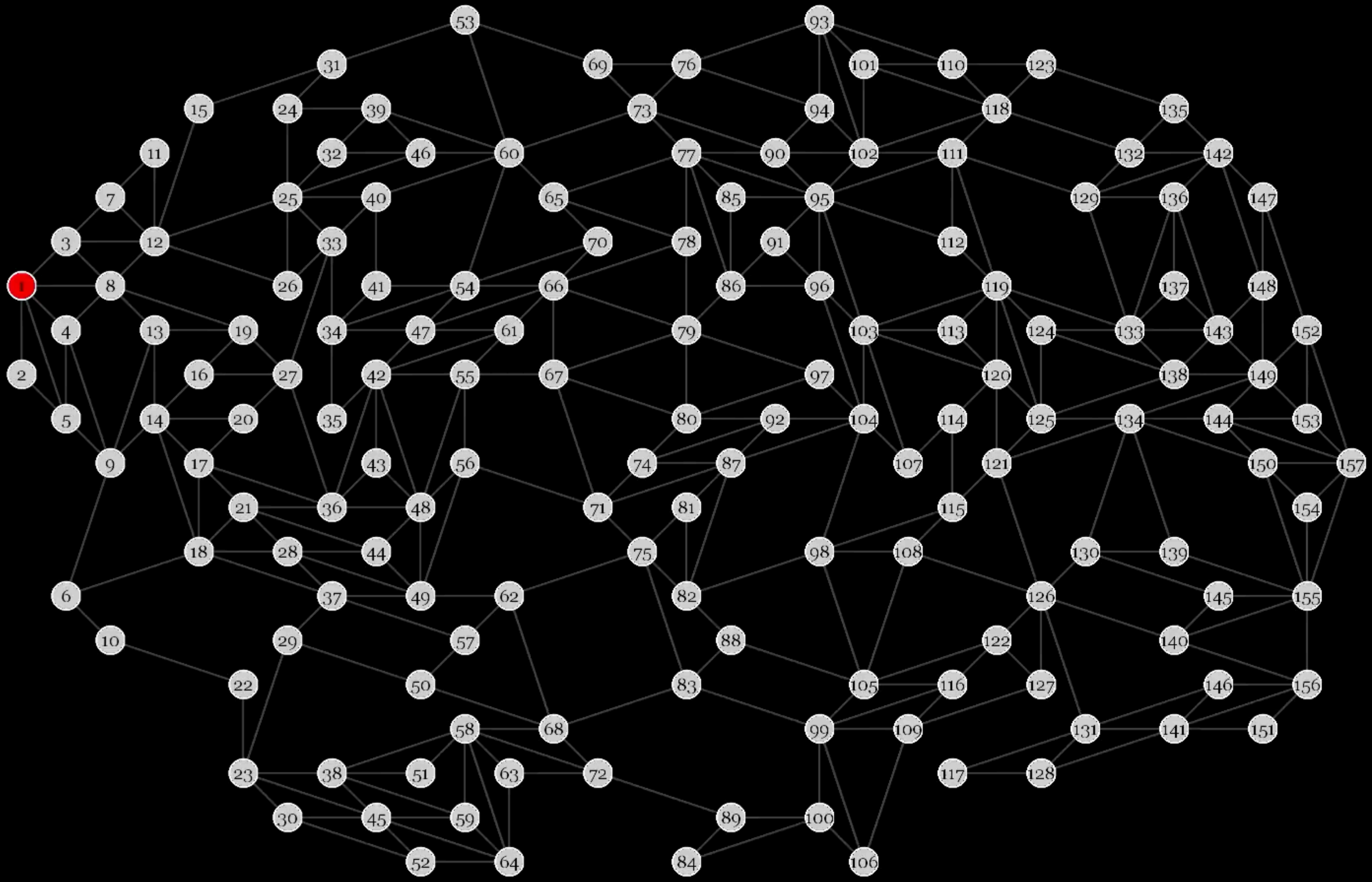
ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an un-direct graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

Credits

image	source	license
Pac–Man Graph	Oatzy	
Pac–Man Game	Old Classic Retro Gaming	
London Tube Map	Transport for London	
London Tube Graph	visualize.org	
Visualizing Friendships	Paul Butler / Facebook	
Twitter Graph	allthingsgraphed.com	
Protein Interaction Graph	Hawing Jeong / KAIST	
PageRank	Wikipedia	public domain
Control Flow Graph	Stack Exchange	
DFS Graph Visualization	Gerry Jenkins	

DFS visualization (by Gerry Jenkins)



https://www.youtube.com/watch?v=NUgMa5coCoE