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3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs (representation)*
- *red–black BSTs (operations)*
- *context*

Symbol table review

implementation	guarantee			ordered ops?	key interface	emoji
	search	insert	delete			
sequential search (unordered list)	n	n	n		`equals()`	☹️
binary search (sorted array)	$\log n$	n	n	✔	`compareTo()`	😕
BST	n	n	n	✔	`compareTo()`	😕
goal	$\log n$	$\log n$	$\log n$	✔	`compareTo()`	😎

Challenge. $O(\log n)$ time in worst case.

optimized for teaching and coding (introduced in COS 226)

This lecture. 2–3 trees and left-leaning red-black BSTs.

co-invented by Bob Sedgewick in the 1970s

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3.3 Balanced Search Trees

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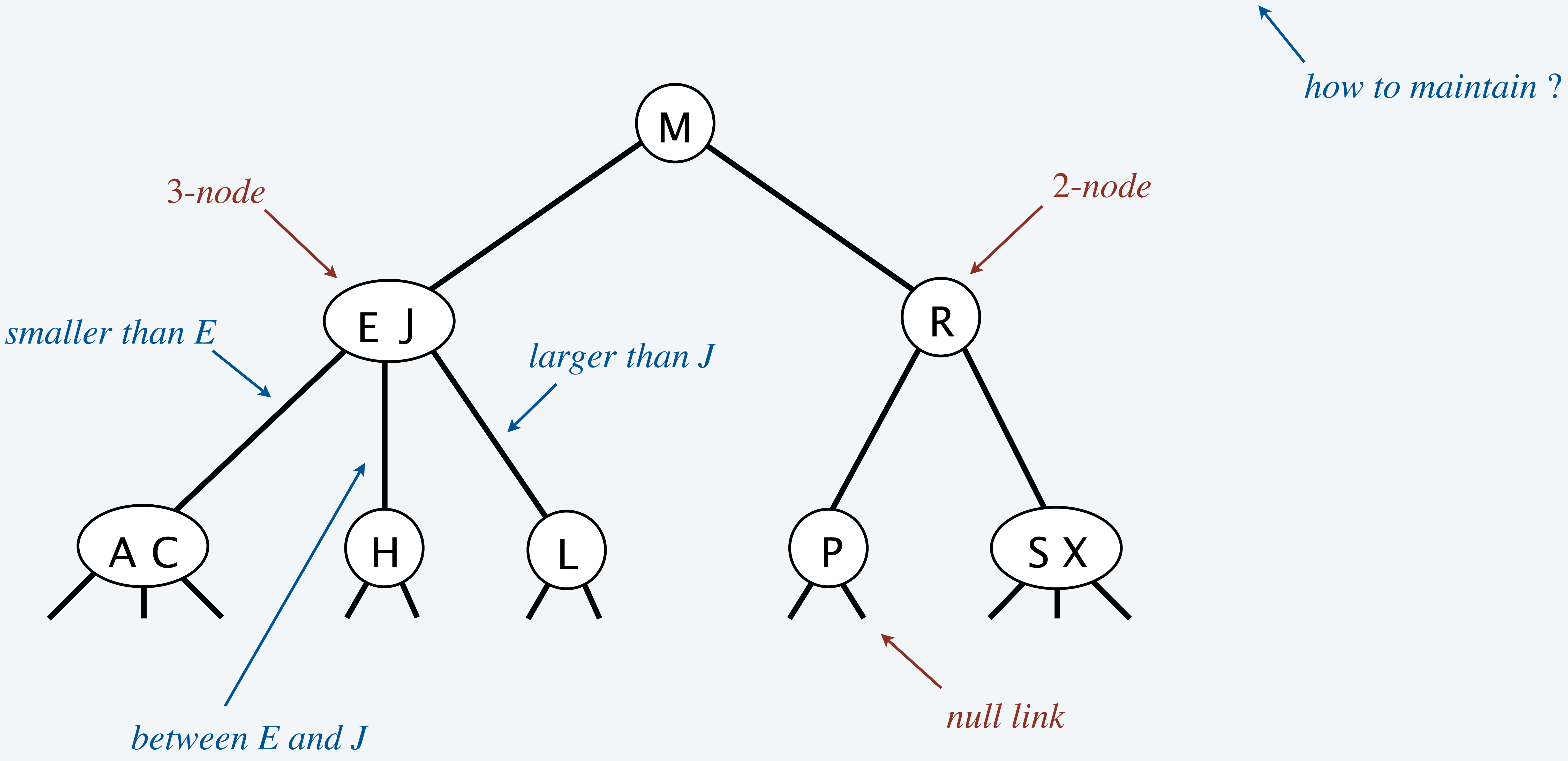
2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from the root to a null link has the same length.

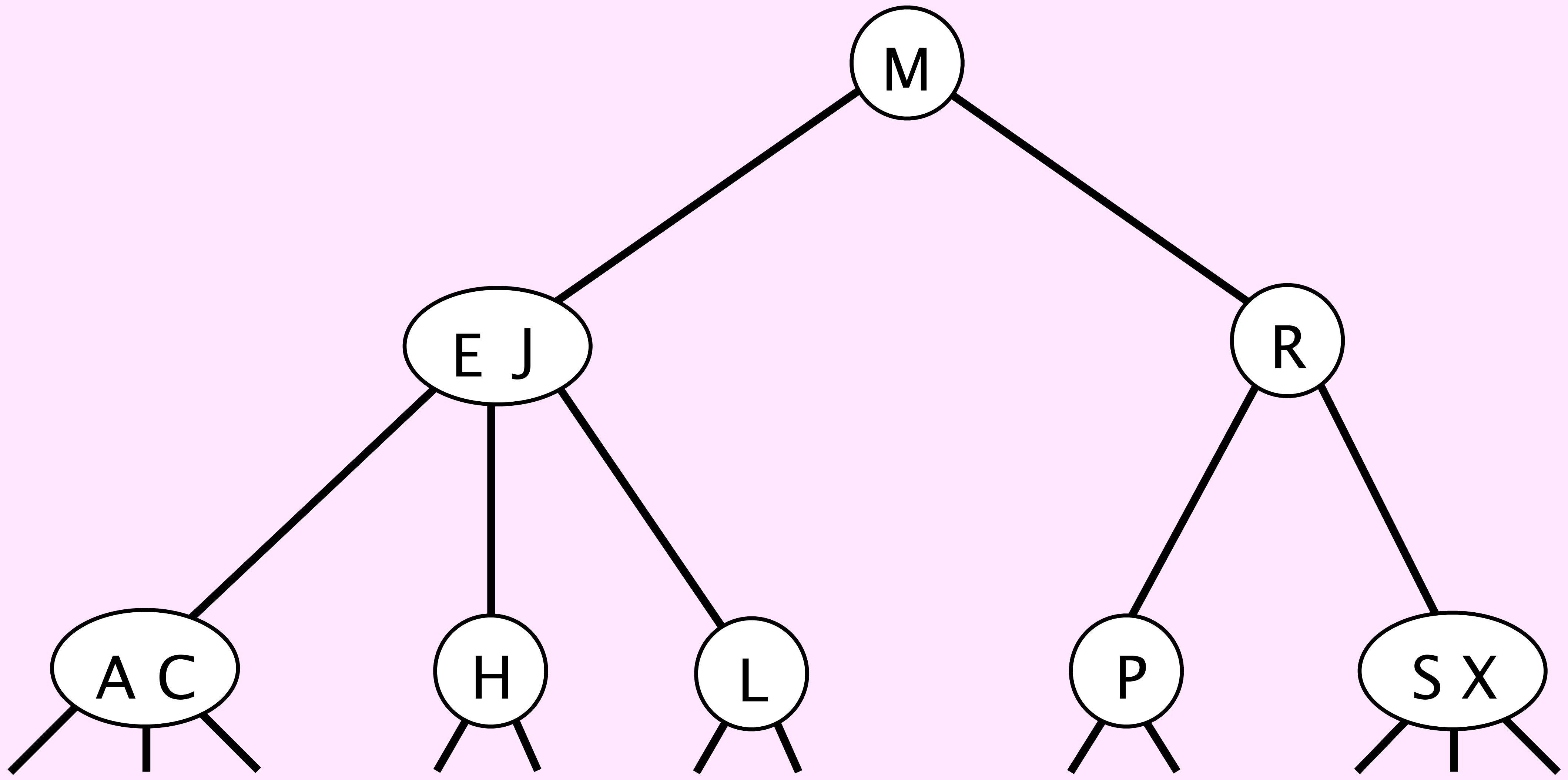


2-3 tree demo

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

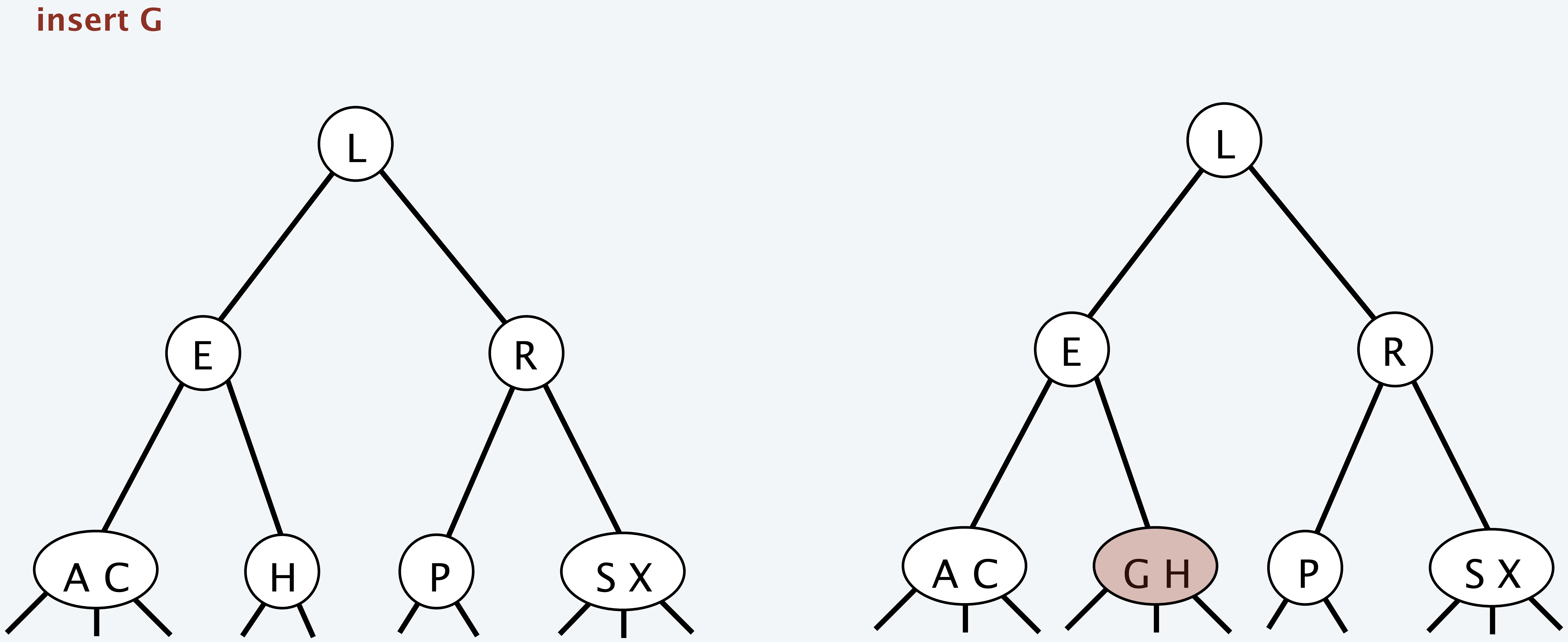
search for H



2-3 tree: insertion

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

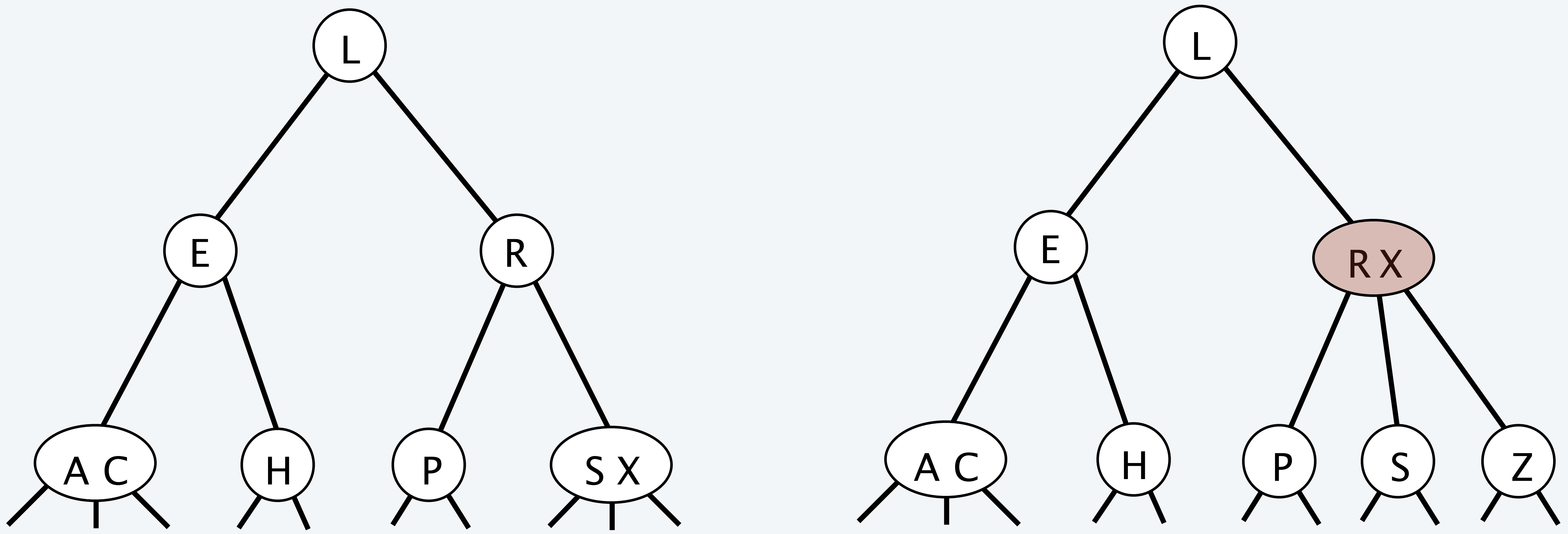


2–3 tree: insertion

Insertion into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

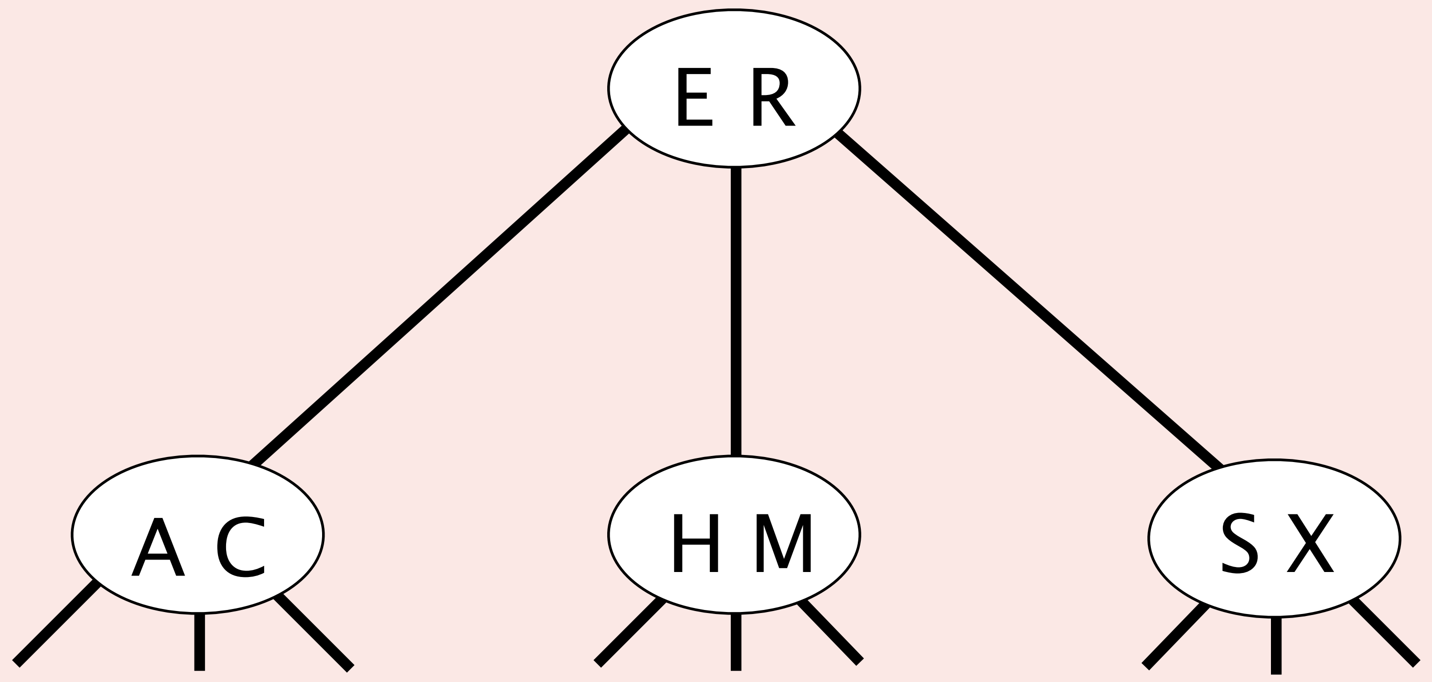
insert Z



Suppose that you insert P into the following 2–3 tree.

What will be the root of the resulting 2–3 tree?

- **A.** E
- **B.** E R
- **C.** M
- **D.** P
- **E.** R



What is the maximum height of a 2–3 tree containing n keys?

A. $\sim \log_3 n$

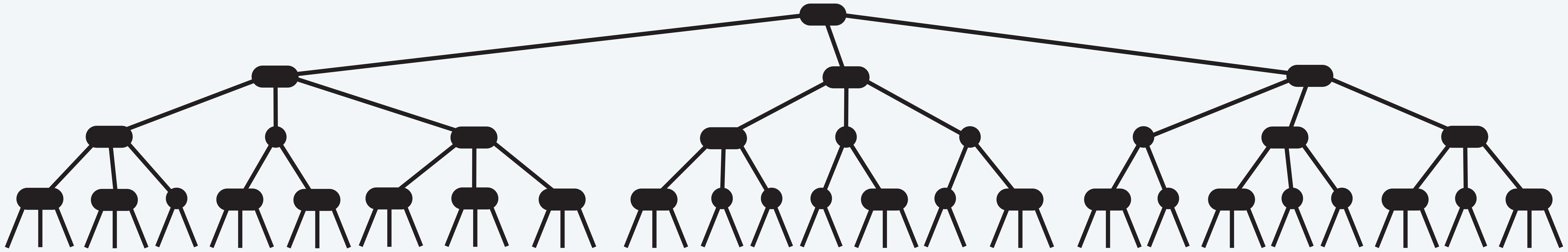
B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

D. $\sim n$

2-3 tree: performance

Perfect balance. Every path from the root to a null link has the same length.



Key property. The height of a 2-3 tree containing n keys is $\Theta(\log n)$.

- Min: $\sim \log_3 n \approx 0.631 \log_2 n$. [all 3-nodes]
- Max: $\sim \log_2 n$. [all 2-nodes]
- Between 18 and 30 for a billion keys.

Bottom line. Search and insert take $\Theta(\log n)$ time in the worst case.

ST implementations: summary

implementation	guarantee			ordered ops?	key interface	emoji
	search	insert	delete			
sequential search (unordered list)	n	n	n		`equals()`	☹️
binary search (sorted array)	$\log n$	n	n	✔	`compareTo()`	😕
BST	n	n	n	✔	`compareTo()`	😕
2-3 trees	$\log n$	$\log n$	$\log n$	✔	`compareTo()`	😎

but hidden constant c is large
(depends upon implementation)

2–3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Might need two compares to move one level down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

fantasy code

```
public void put(Key key, Value val) {
   Node x = root;
   while (x.getTheCorrectChild(key) != null) {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it (see COS 326!), but there's a better way.

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3.3 Balanced Search Trees

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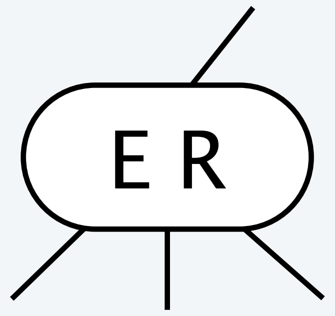
- red–black BSTs (representation)
- red–black BSTs (operations)
- context

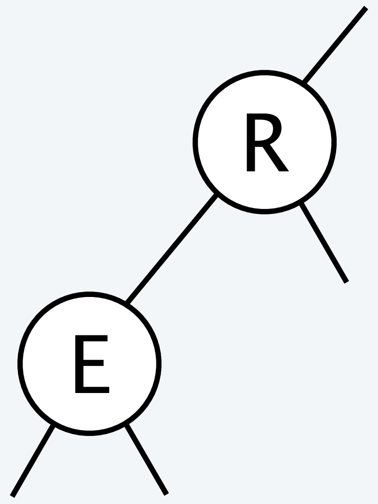
How to implement 2-3 trees as binary search trees?

Challenge. How to represent a 3 node?



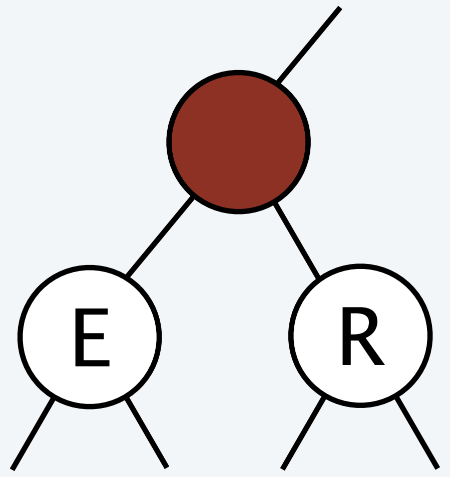
Approach 1. Two BST nodes.

- No way to tell a 3-node from two 2-nodes.
- Can't (uniquely) map from BST back to 2-3 tree.



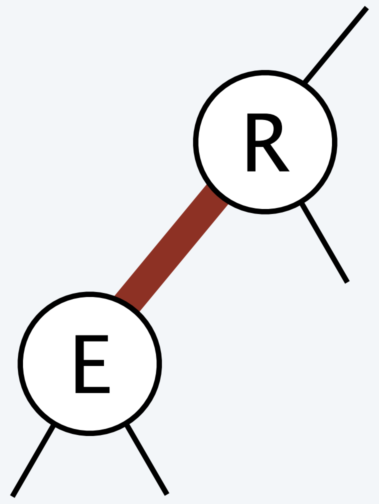
Approach 2. Two BST nodes, plus red "glue" node.

- Wastes space for extra node.
- Messy code.



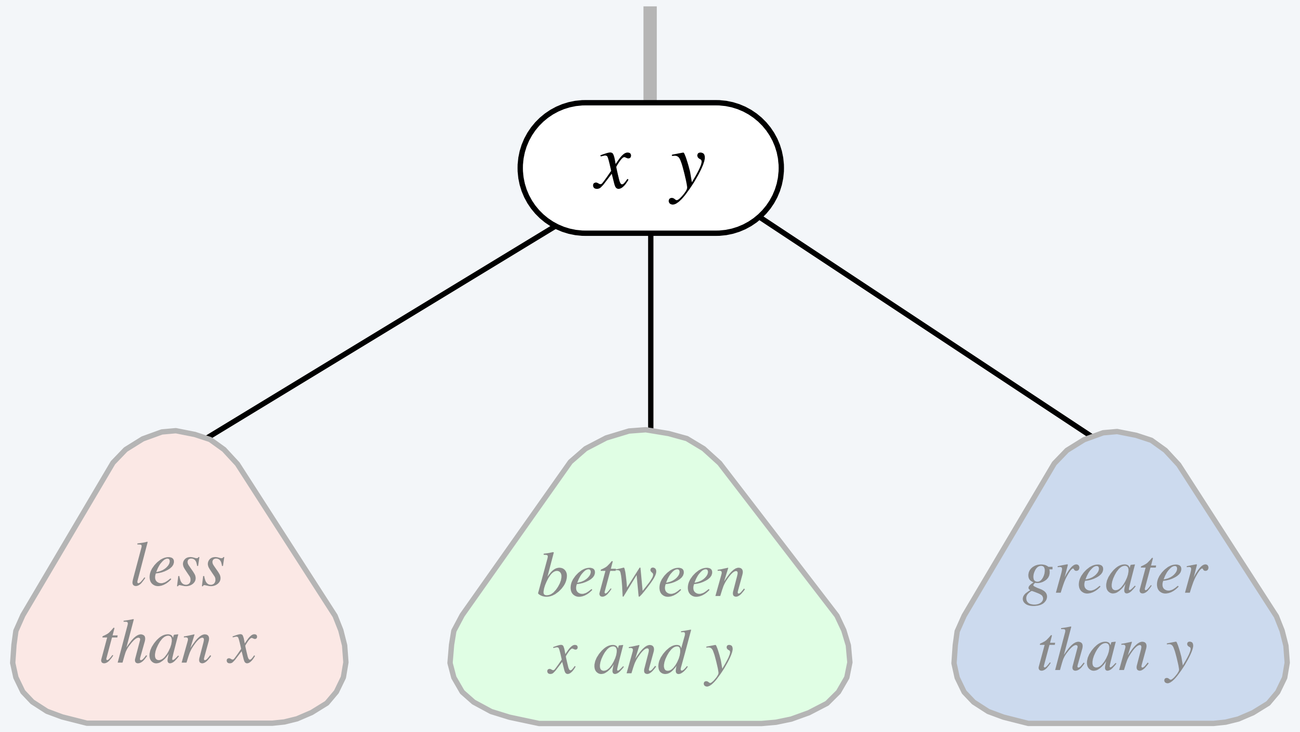
Approach 3. Two BST nodes, with red "glue" link.

- Widely used in practice.
- Arbitrary restriction: red links lean left.

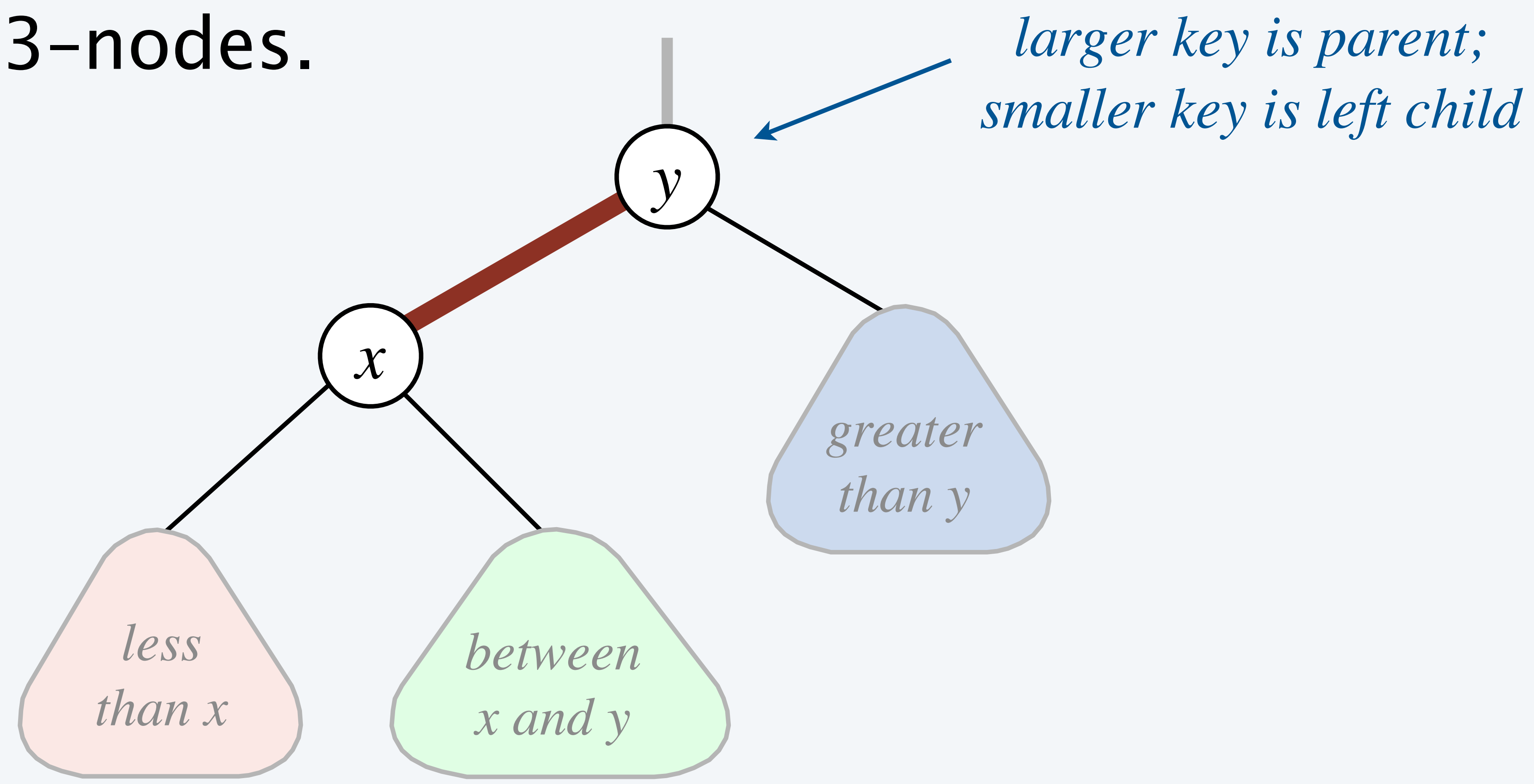


Left-leaning red–black BSTs

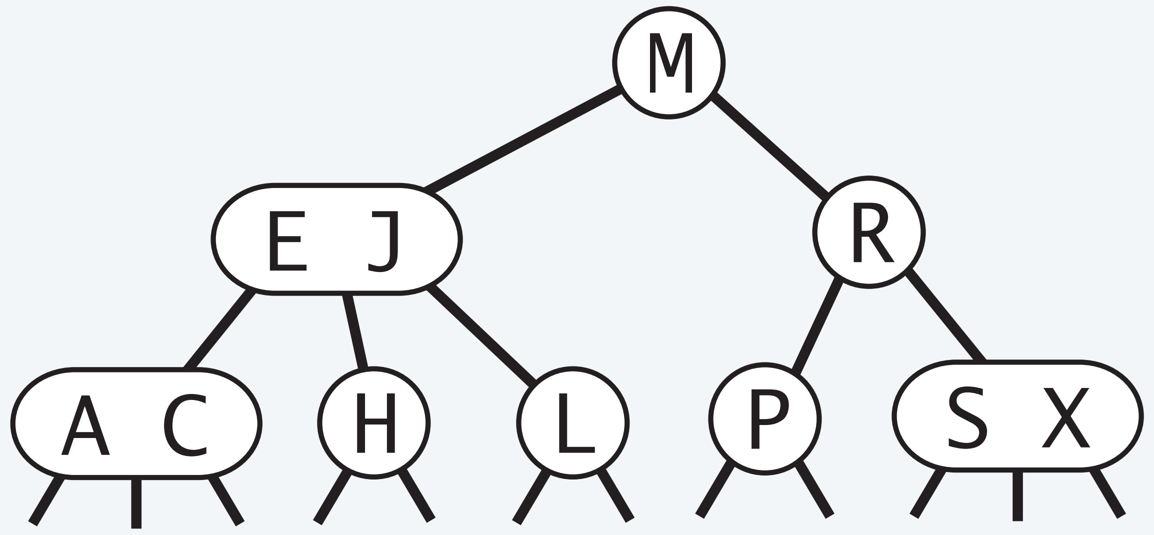
1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning red links as "glue" for 3-nodes.



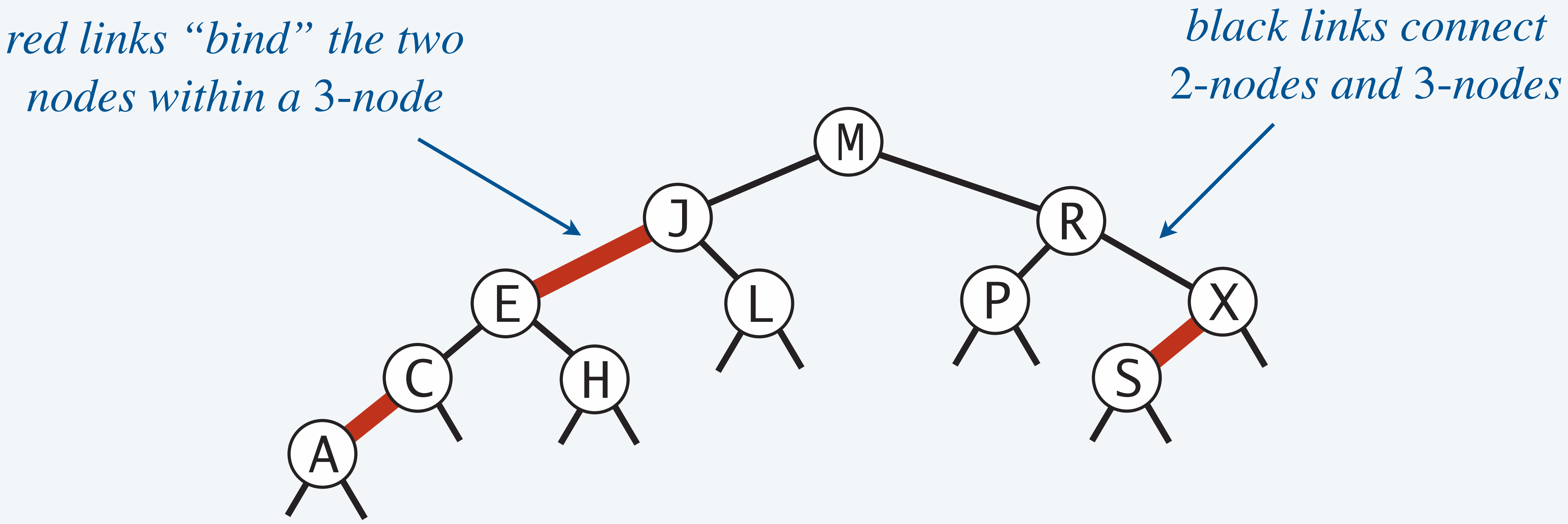
3-node in a 2–3 tree



two nodes in the corresponding red–black BST



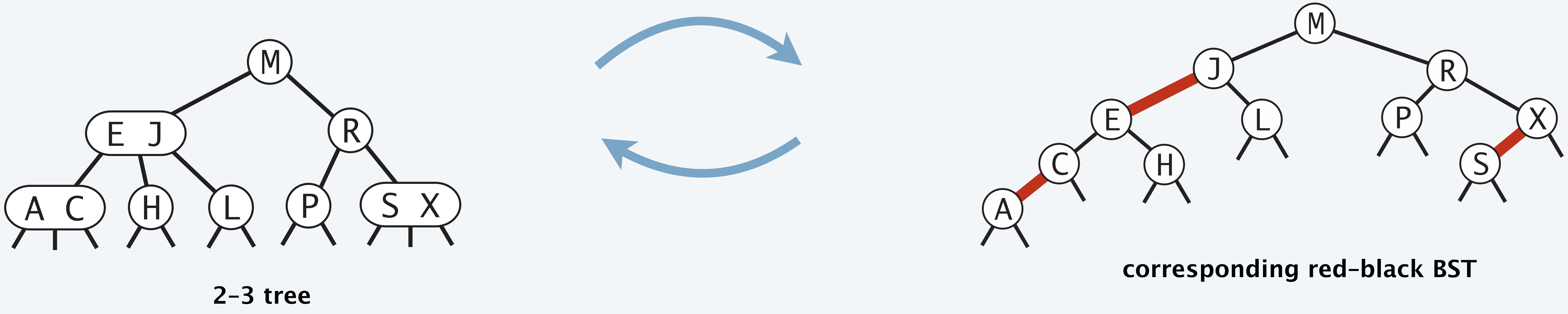
2–3 tree



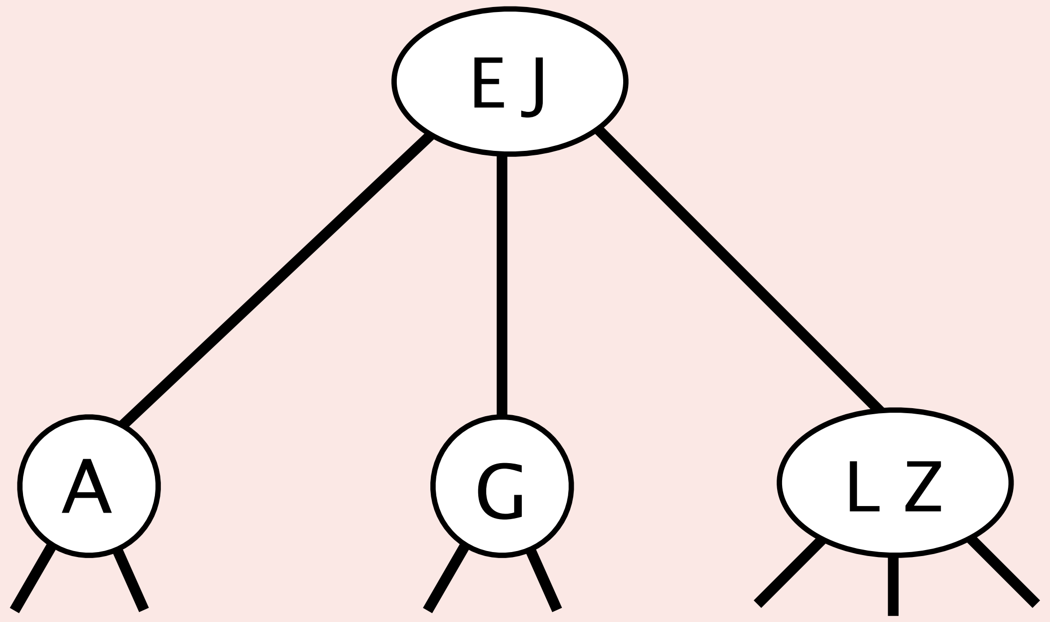
corresponding red–black BST

Left-leaning red-black BSTs

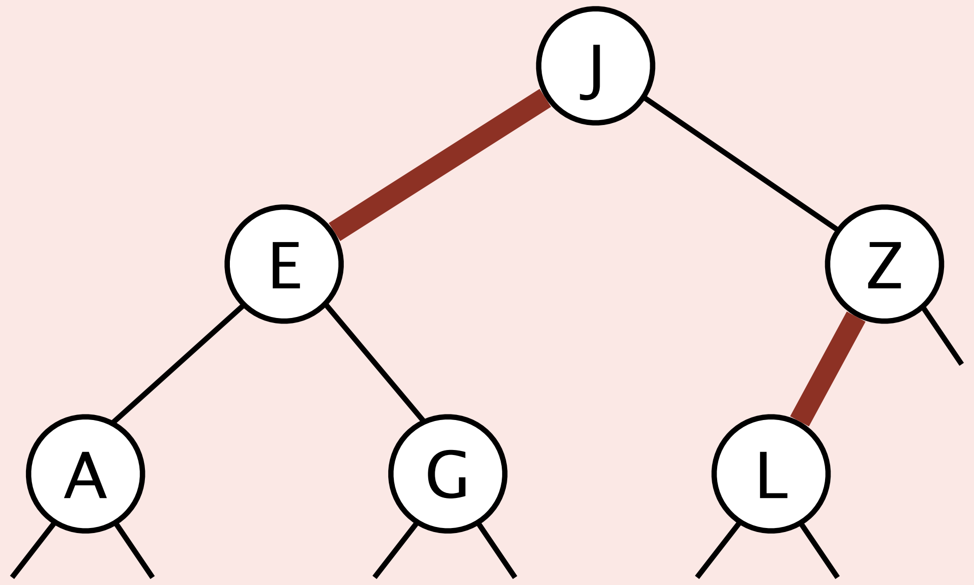
Key property. 1-1 correspondence between 2-3 trees and LLRB trees.



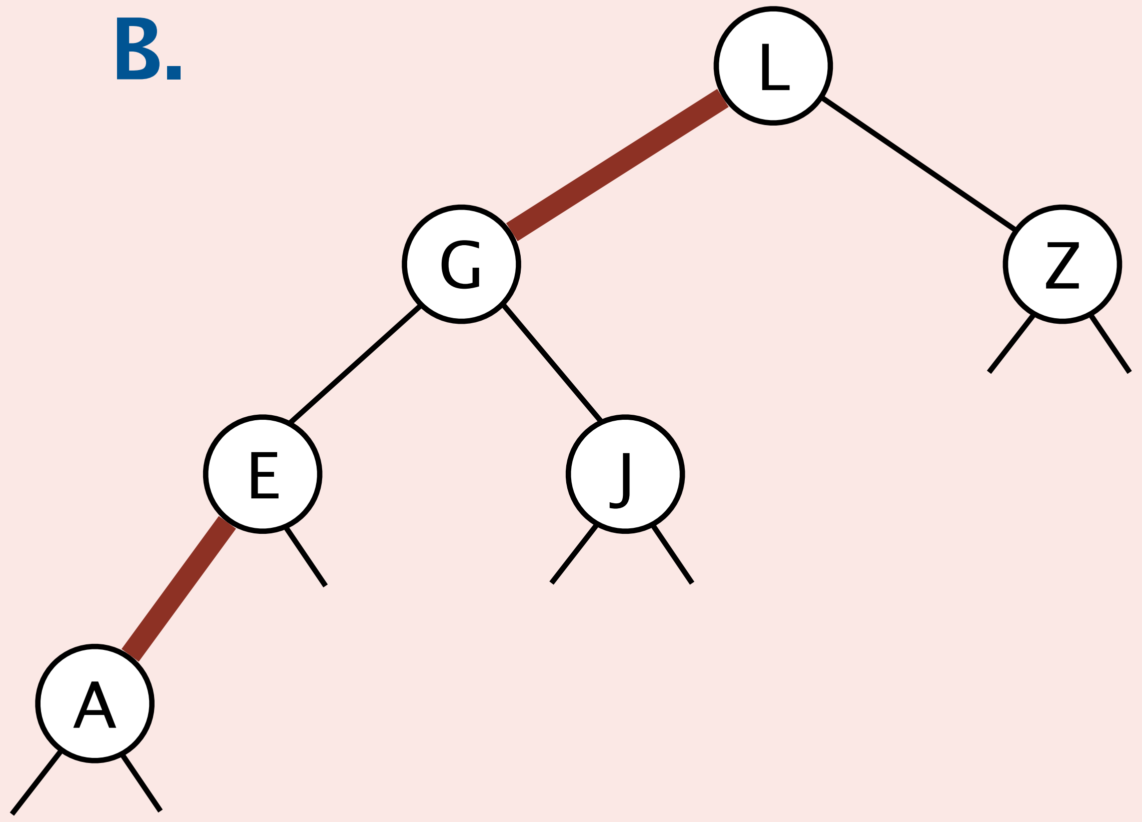
Which LLRB tree corresponds to the following 2-3 tree?



A.



B.



C. Both A and B.

D. Neither A nor B.

An equivalent definition of LLRB trees (without reference to 2–3 trees)

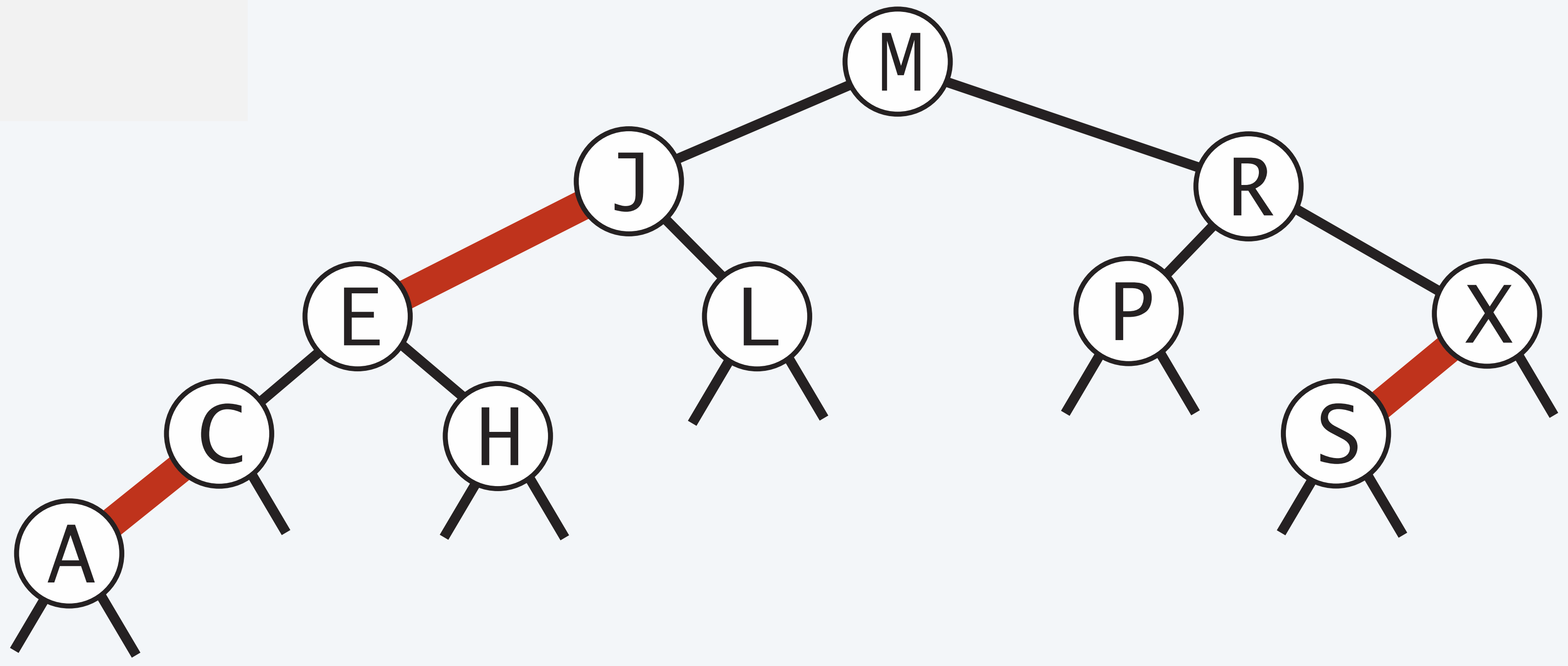
symmetric order

Def. A red–black BST is a BST such that:

- No node has two red links connected to it.
- Red links lean left.
- Every path from root to null link has the same number of black links.

color invariants

"perfect black balance"

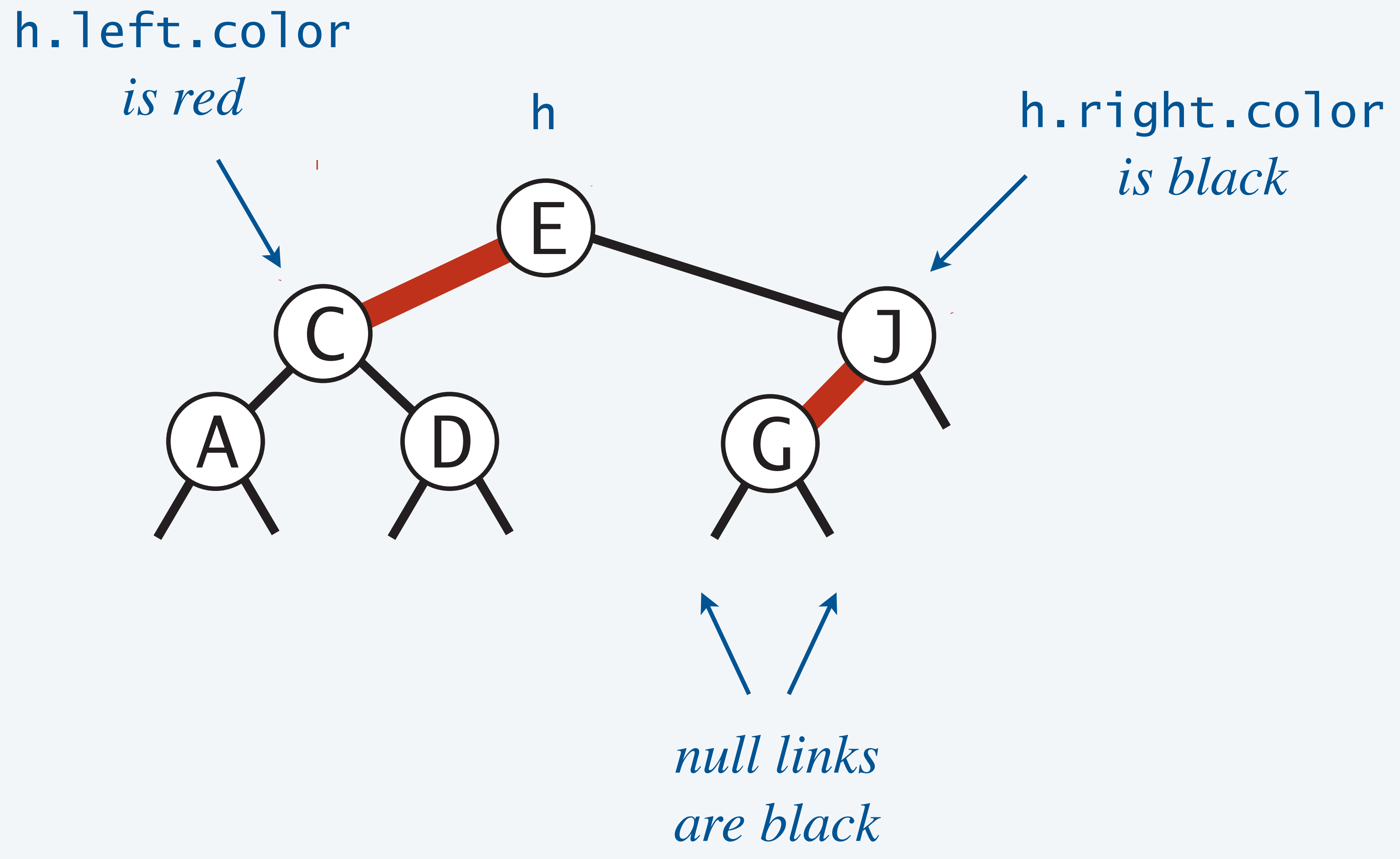


Red-black BST representation

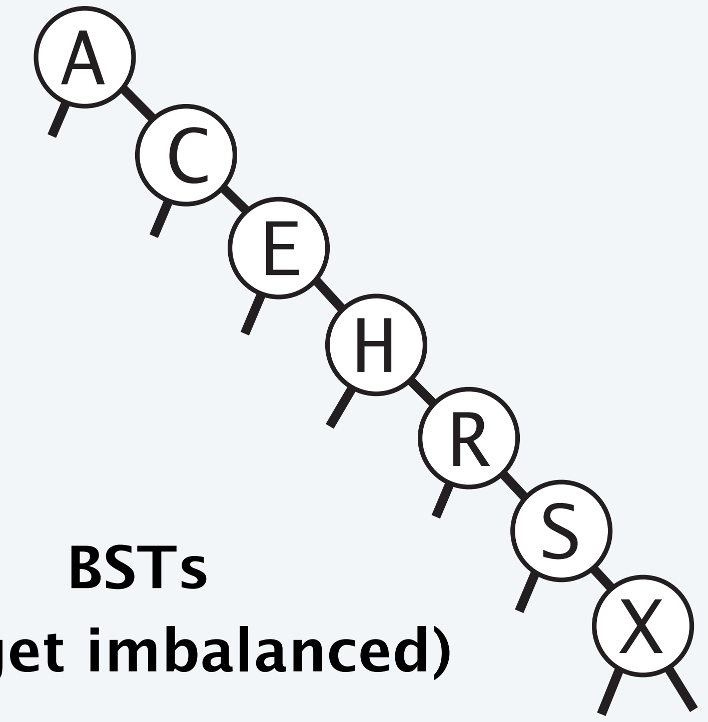
Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in child nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node {
   private Key key;
   private Value val;
   private Node left, right;
   private boolean color;     ← color of parent link
}

private boolean isRed(Node h) {
   if (h == null) return false;
   return h.color == RED;
}
```

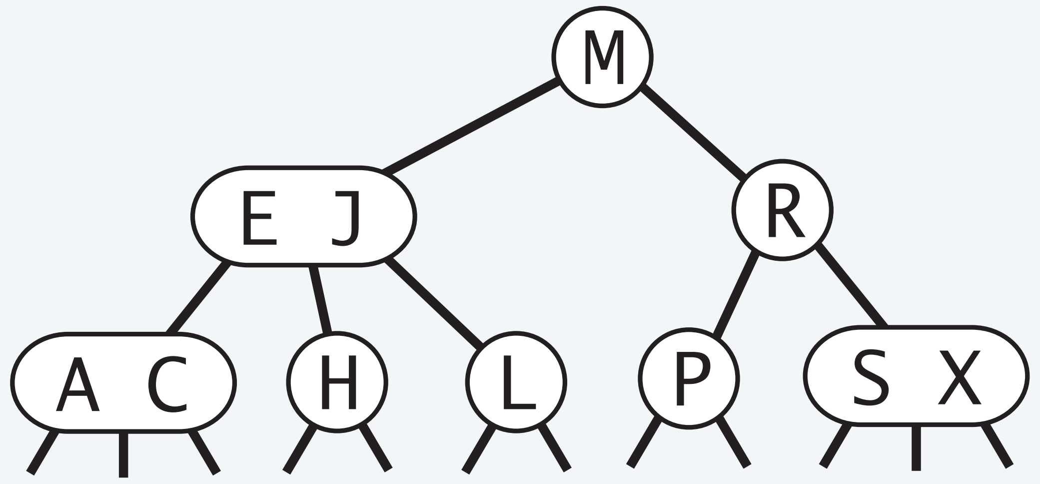
null links are black



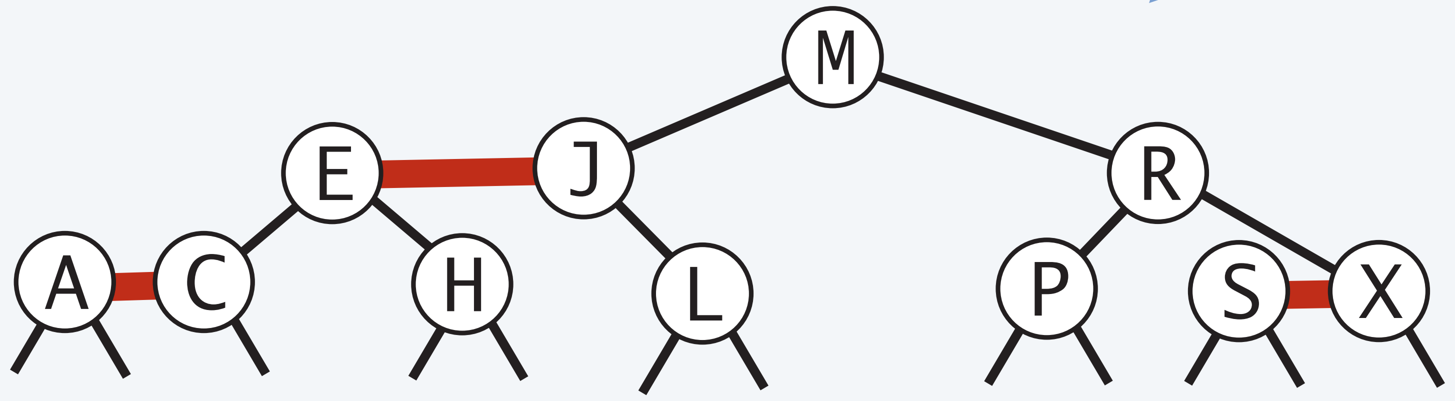
Review: the road to LLRB trees



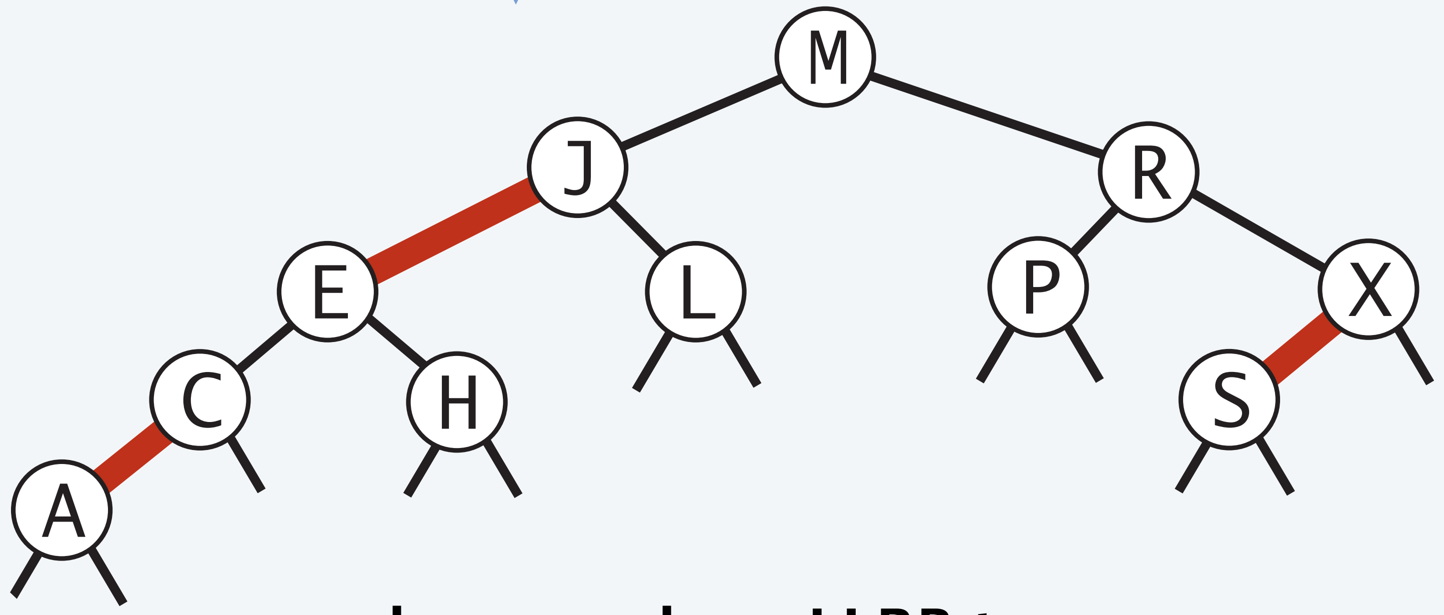
BSTs
(can get imbalanced)



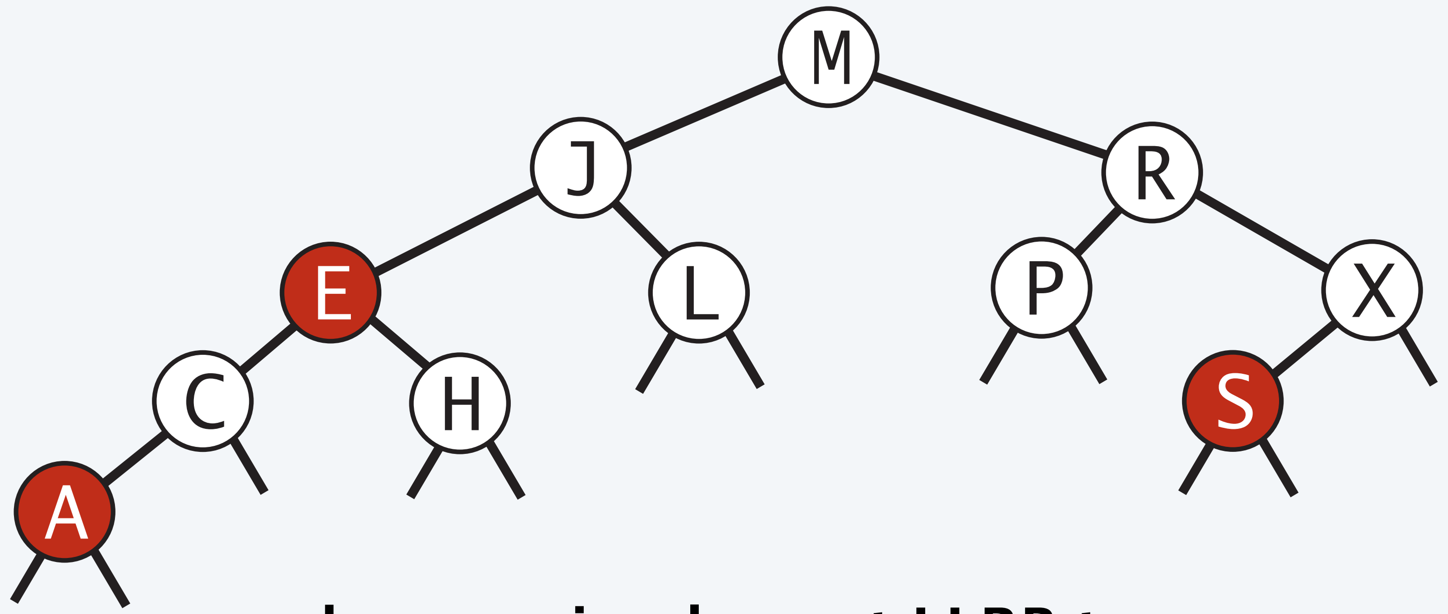
2-3 trees
(balanced but awkward to implement)



3-nodes "glued" together with red links



how we draw LLRB trees
(color in links)



how we implement LLRB trees
(color in nodes)

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- red–black BSTs (operations)
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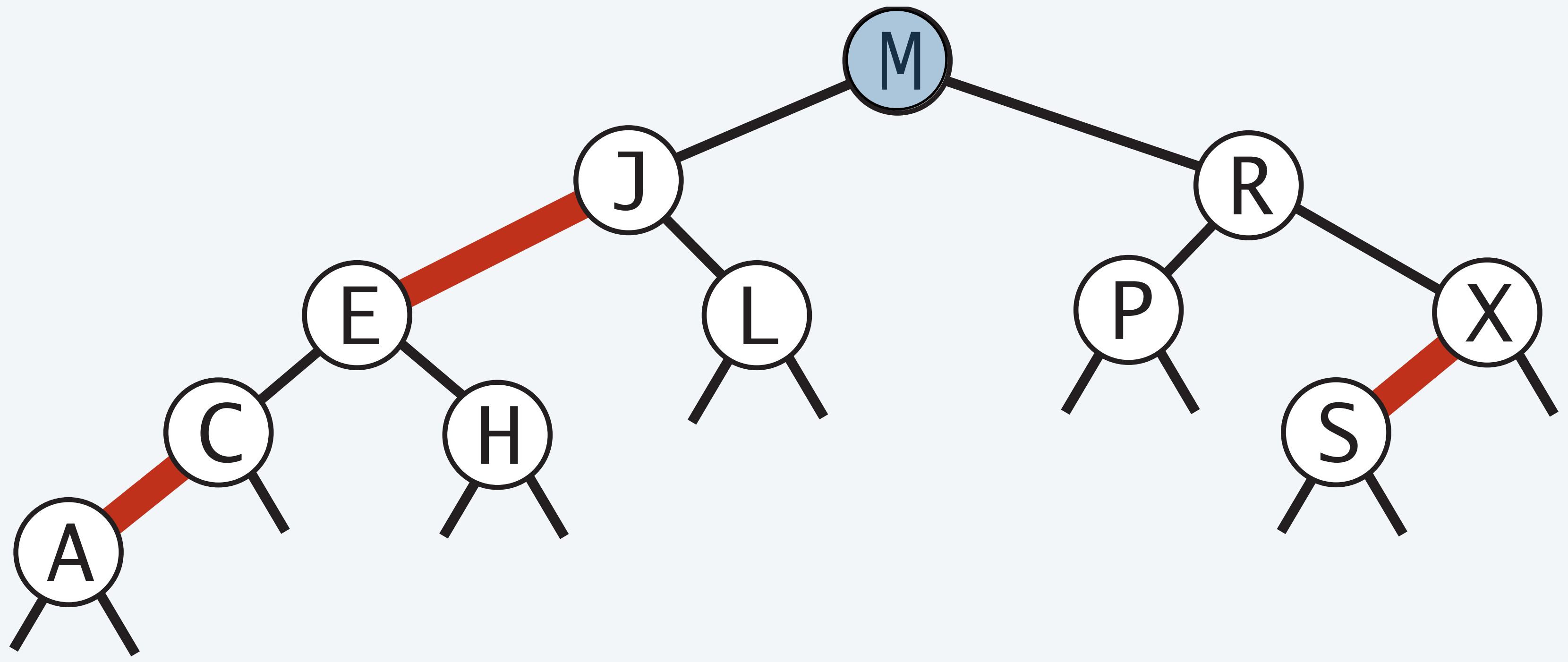
Search in a red-black BST

Observation. Red-black BSTs are BSTs $\Rightarrow$ search is the same as for BSTs (ignore color).

but runs faster (because of better balance)

```
public Value get(Key key) {
   Node x = root;
   while (x != null) {
      int cmp = key.compareTo(x.key);
      if      (cmp < 0) x = x.left;
      else if (cmp > 0) x = x.right;
      else return x.val;
   }
   return null;
}
```



Remark. Many other operations (iteration, floor, rank, selection) are also identical.

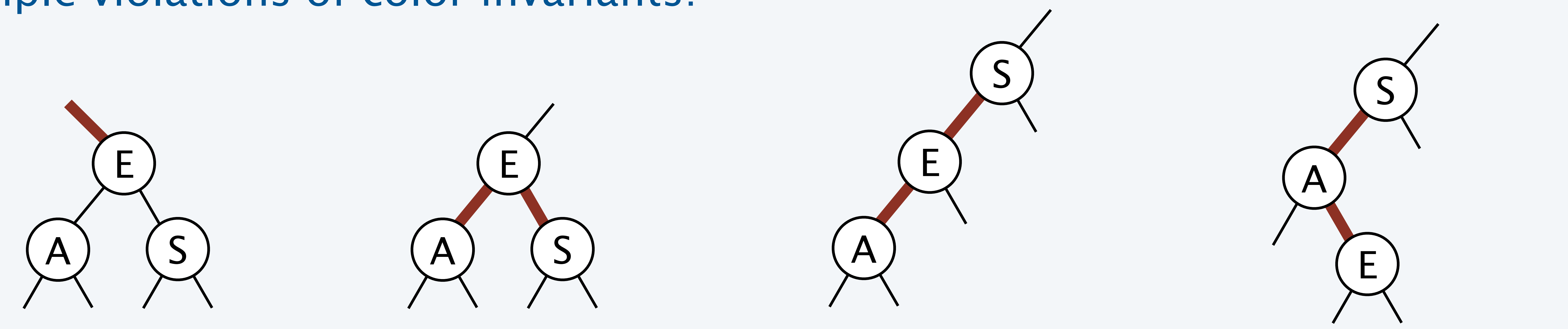
Insertion into a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

During internal operations, maintain:

- Symmetric order.
- Perfect black balance.
- [but not necessarily color invariants]

Example violations of color invariants:



right-leaning red link

two red children (a temporary 4-node)

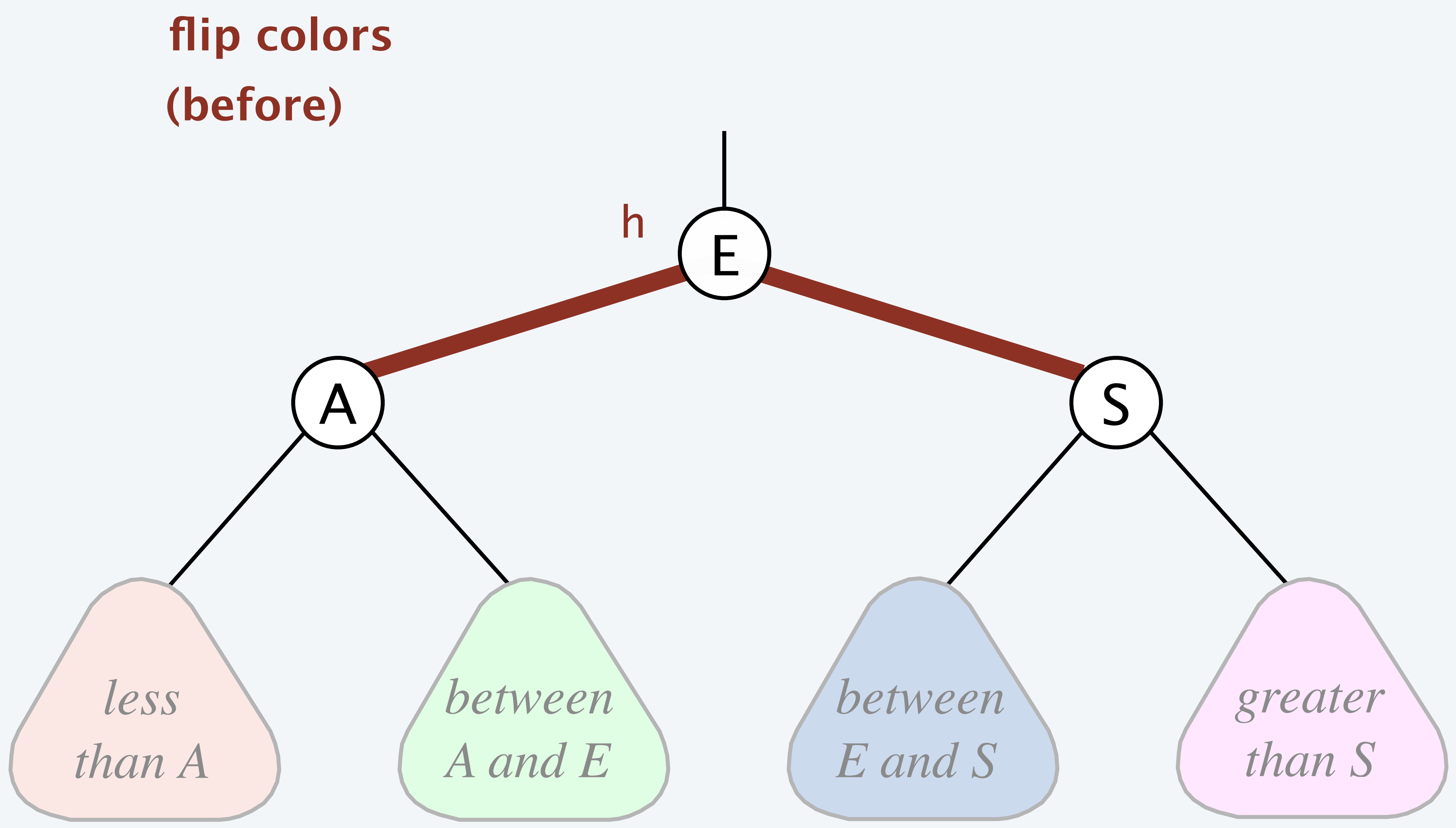
left-left red (a temporary 4-node)

left-right red (a temporary 4-node)

To restore color invariants: perform color flips and rotations.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

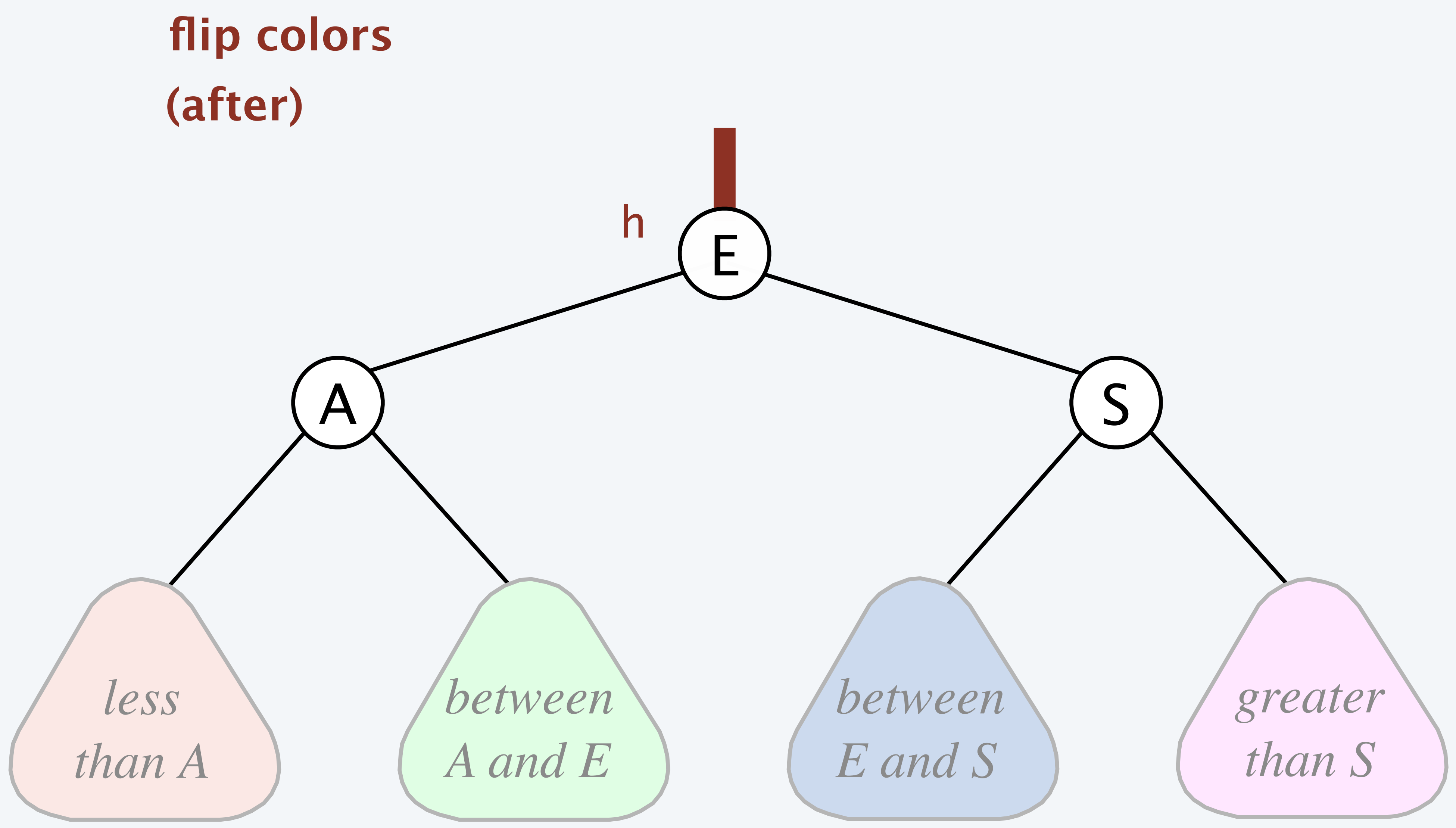



```
private void flipColors(Node h) {
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red–black BST operations

Color flip. Recolor to split a (temporary) 4-node.




```
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   h.left.color = BLACK;
   h.right.color = BLACK;
}
```
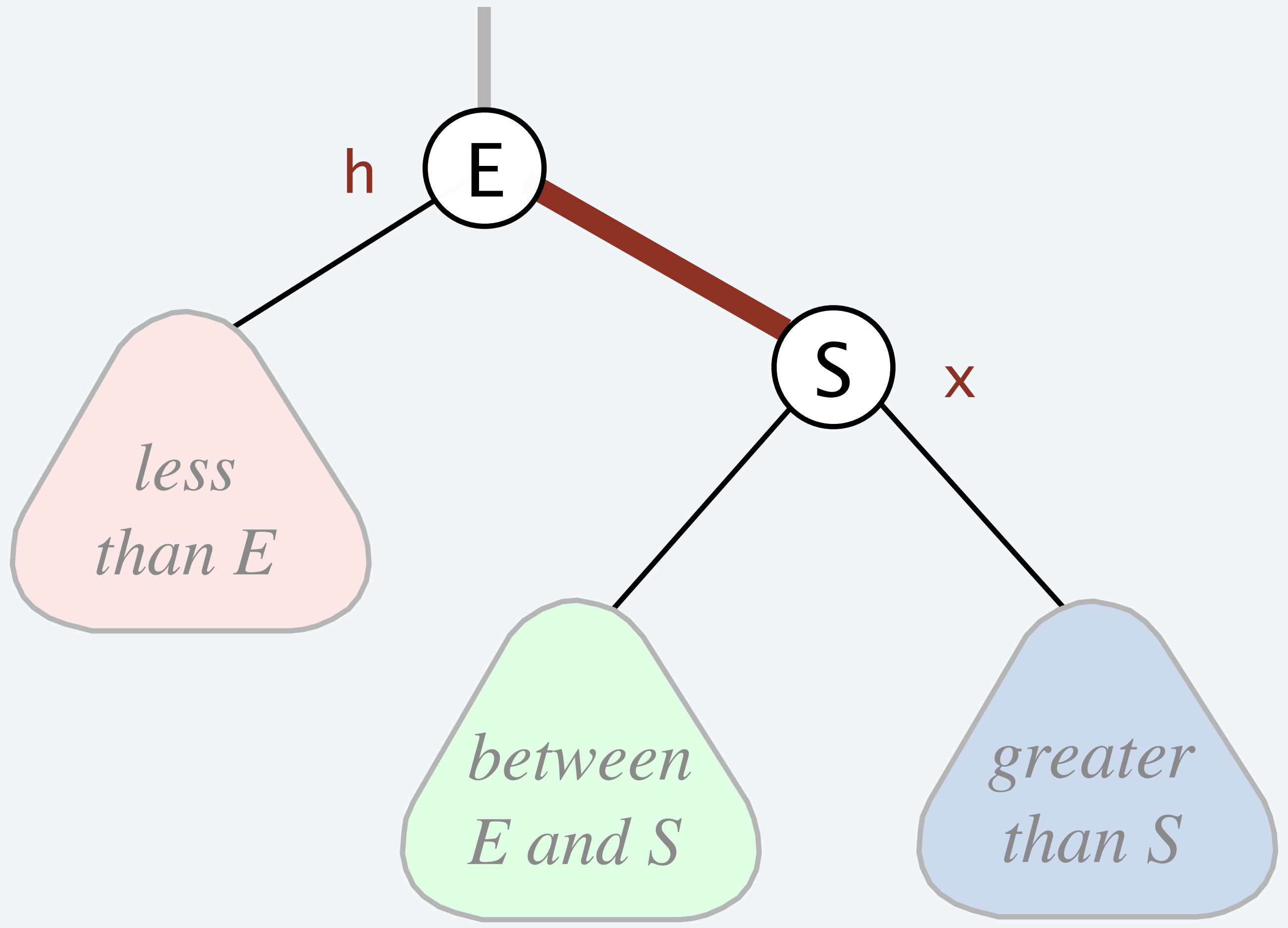

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

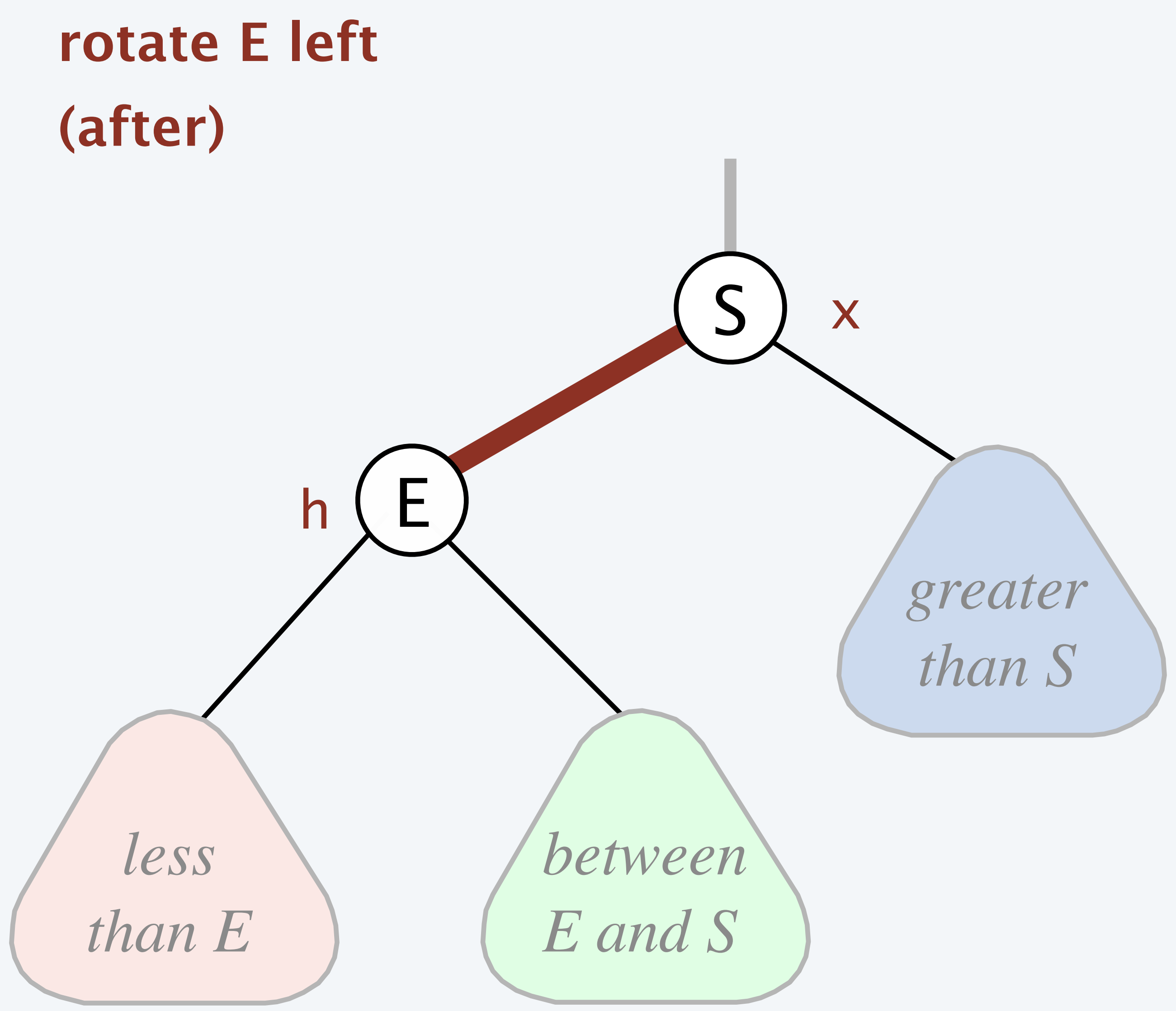



```
private Node rotateLeft(Node h) {
   assert !isRed(h.left);
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red–black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

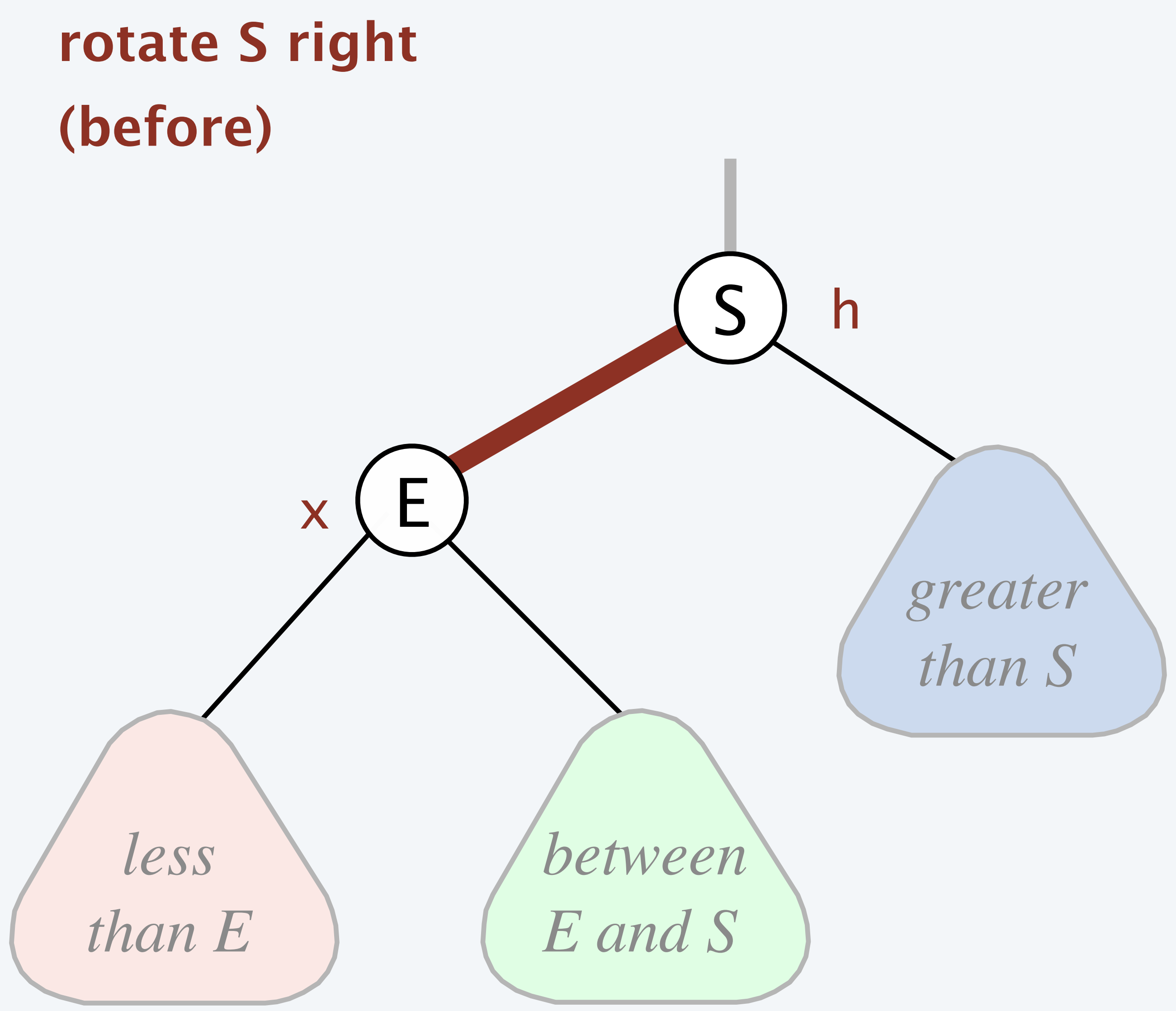



```
private Node rotateLeft(Node h) {
   assert !isRed(h.left);
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

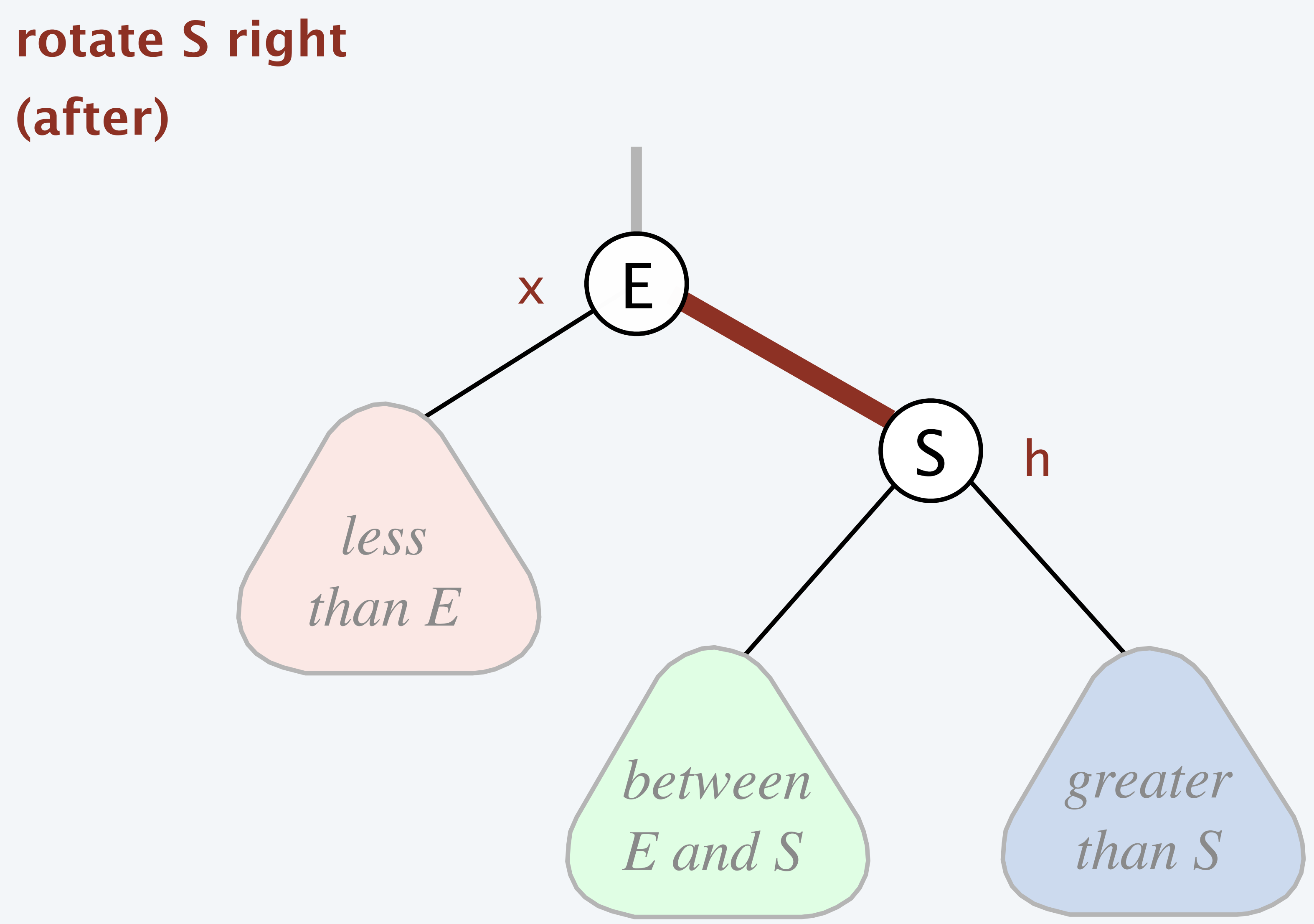



```
private Node rotateRight(Node h) {
   assert isRed(h.left);
   assert !isRed(h.right);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

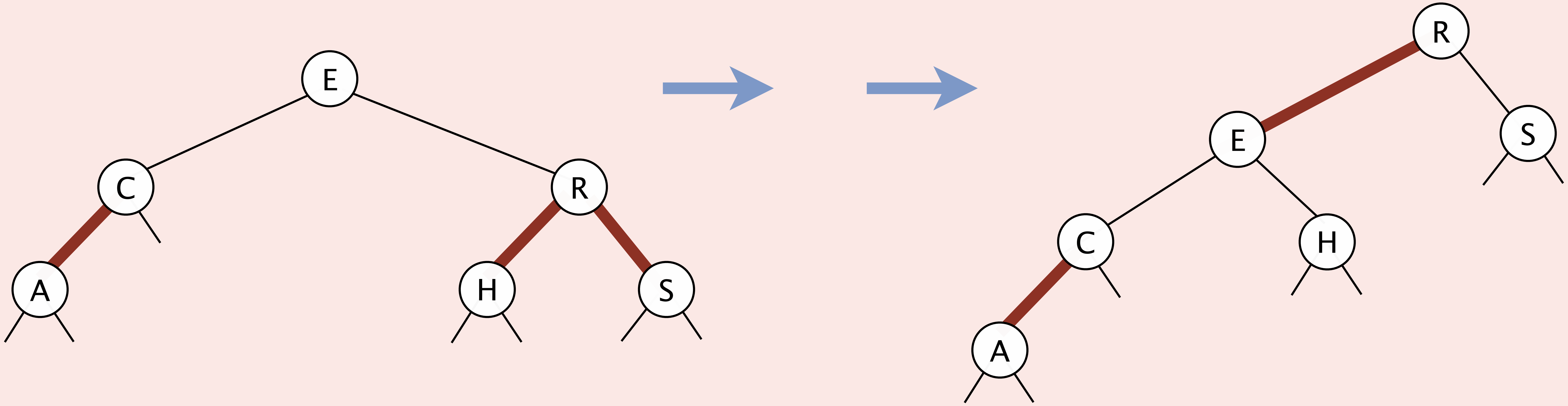
Right rotation. Orient a left-leaning red link to (temporarily) lean right.




```
private Node rotateRight(Node h) {
   assert isRed(h.left);
   assert !isRed(h.right);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Which sequence of elementary operations transforms the red-black BST at left to the one at right?



A. Color flip E; left rotate R.

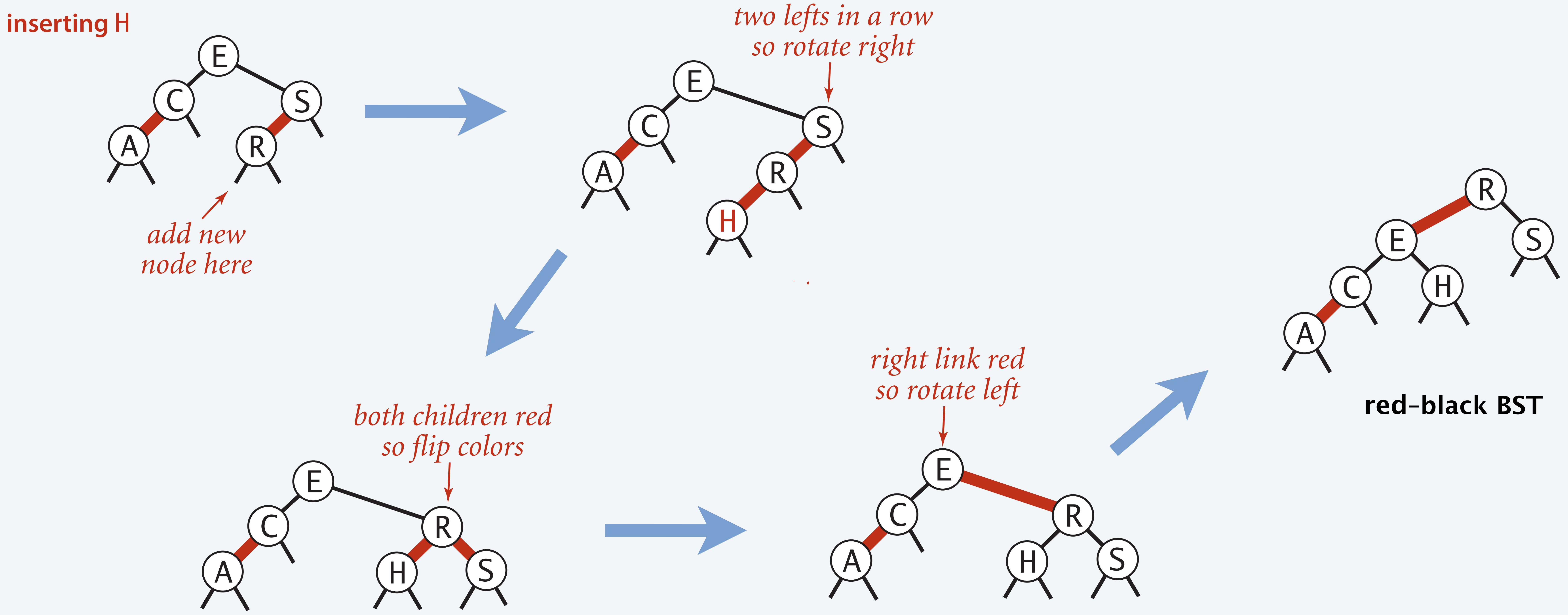
B. Color flip R; left rotate E.

C. Color flip R; left rotate R.

D. Color flip R; right rotate E.

Insertion into a LLRB tree

- Do standard BST insert and color new link red. ← *to preserve symmetric order and perfect black balance*
- Repeat up the tree until color invariants restored:
  - two left red links in a row? ⇒ rotate right
  - left and right links both red? ⇒ color flip
  - only right link red? ⇒ rotate left



Insertion into a LLRB tree

- Do standard BST insert and color new link red.
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  - two left red links in a row? ⇒ rotate right
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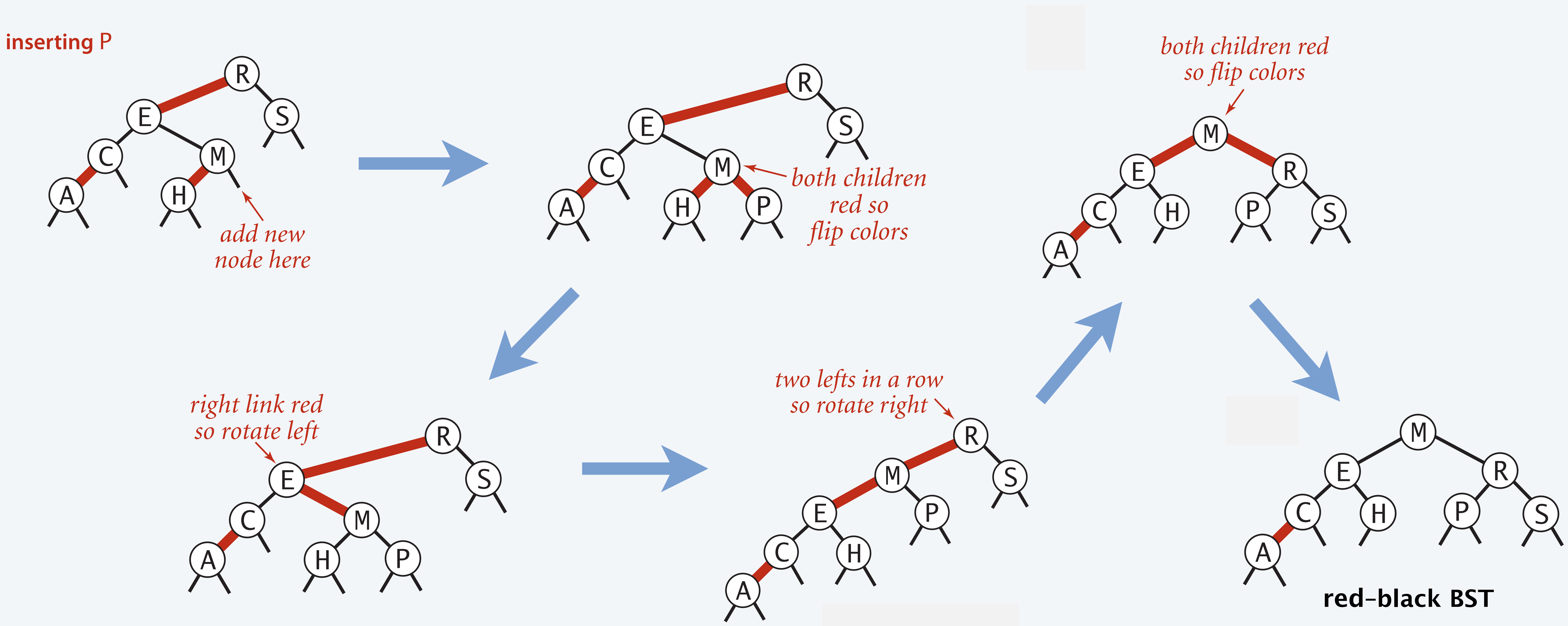
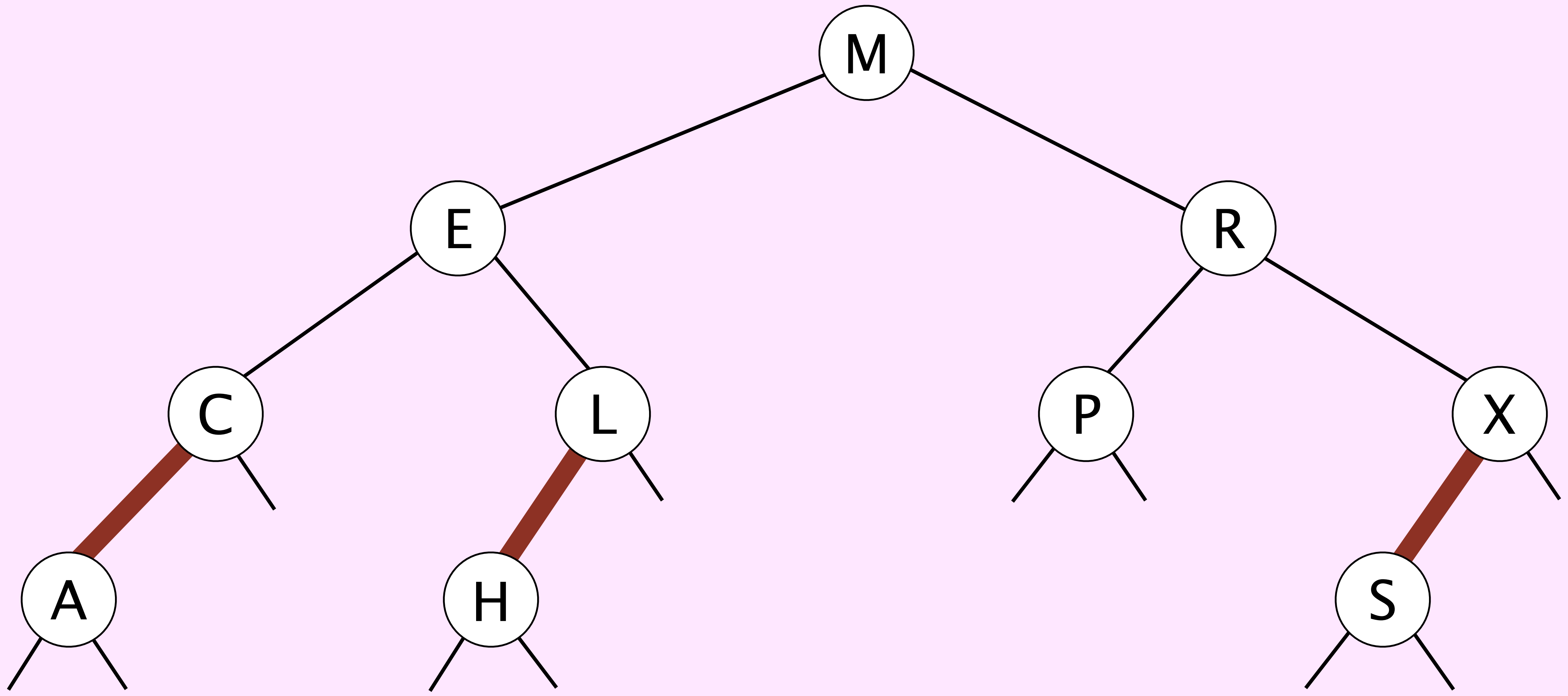


Red-black BST construction demo

insert S E A R C H X M P L



Insertion into a LLRB tree: Java implementation

- Do standard BST insert and color new link red.
- Repeat up the tree until color invariants restored:
  - only right link red? ⇒ rotate left
  - two left red links in a row? ⇒ rotate right
  - left and right links both red? ⇒ color flip

```
private Node put(Node h, Key key, Value val) {
   if (h == null) return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if      (cmp < 0) h.left  = put(h.left,  key, val);
   else if (cmp > 0) h.right = put(h.right, key, val);
   else h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

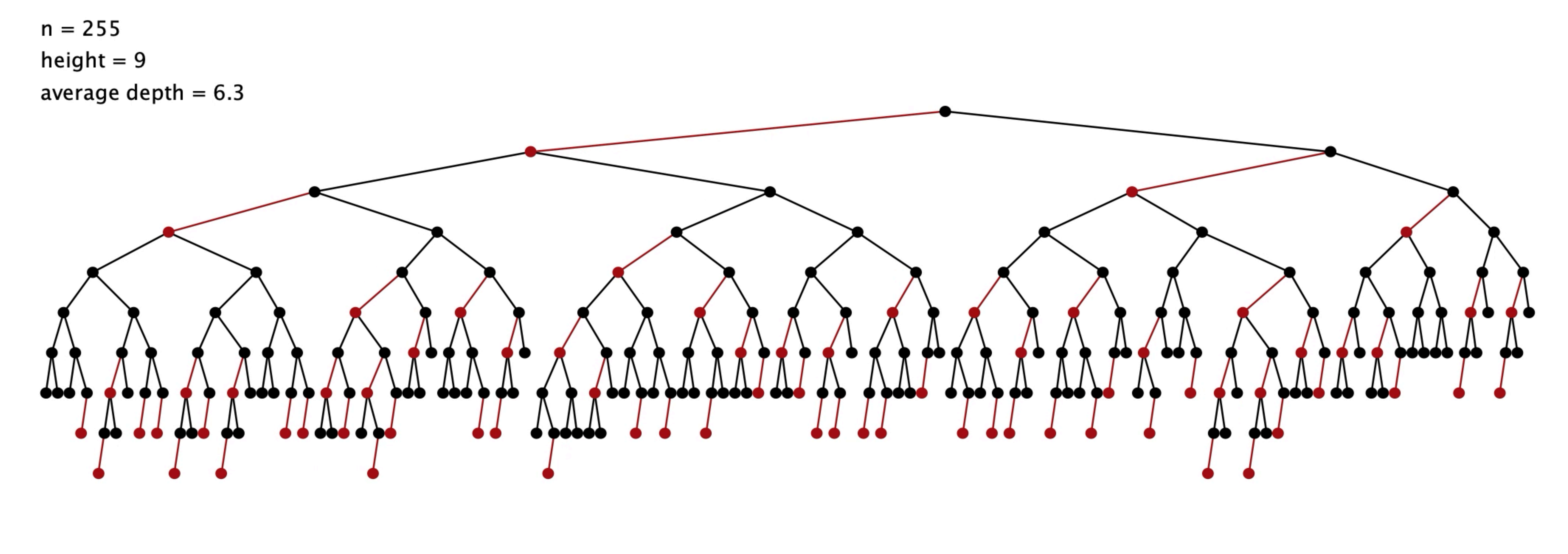
insert at bottom (and color it red)

each method that changes the tree shape returns the root of the resulting subtree

restore color invariants

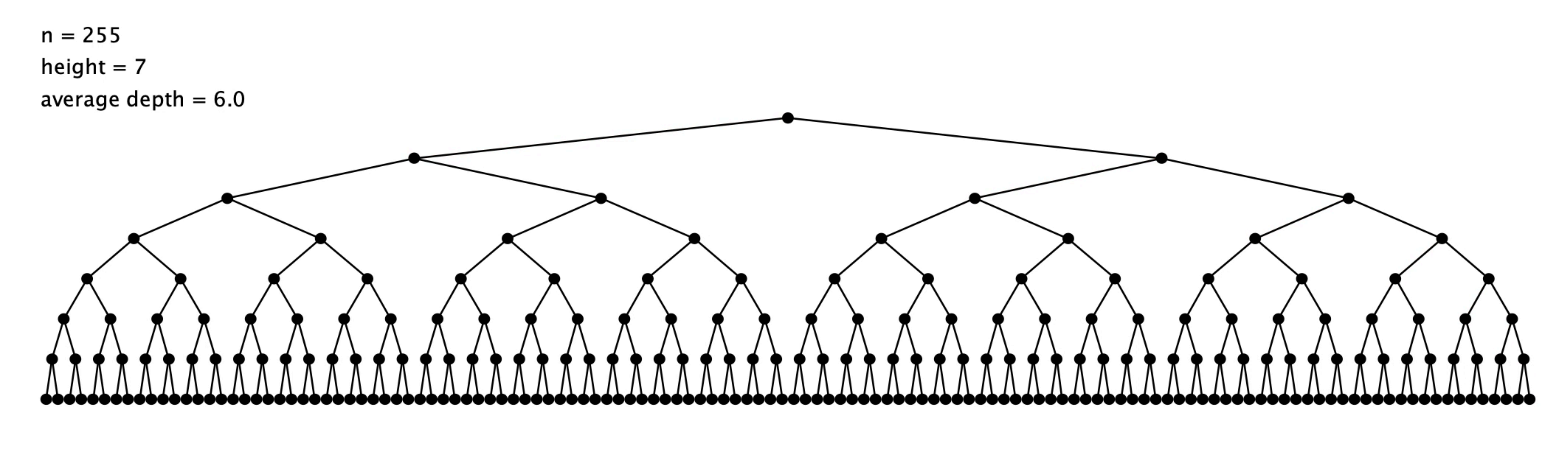
only a few extra lines of code provides near-perfect balance

Insertion into a LLRB tree: visualization



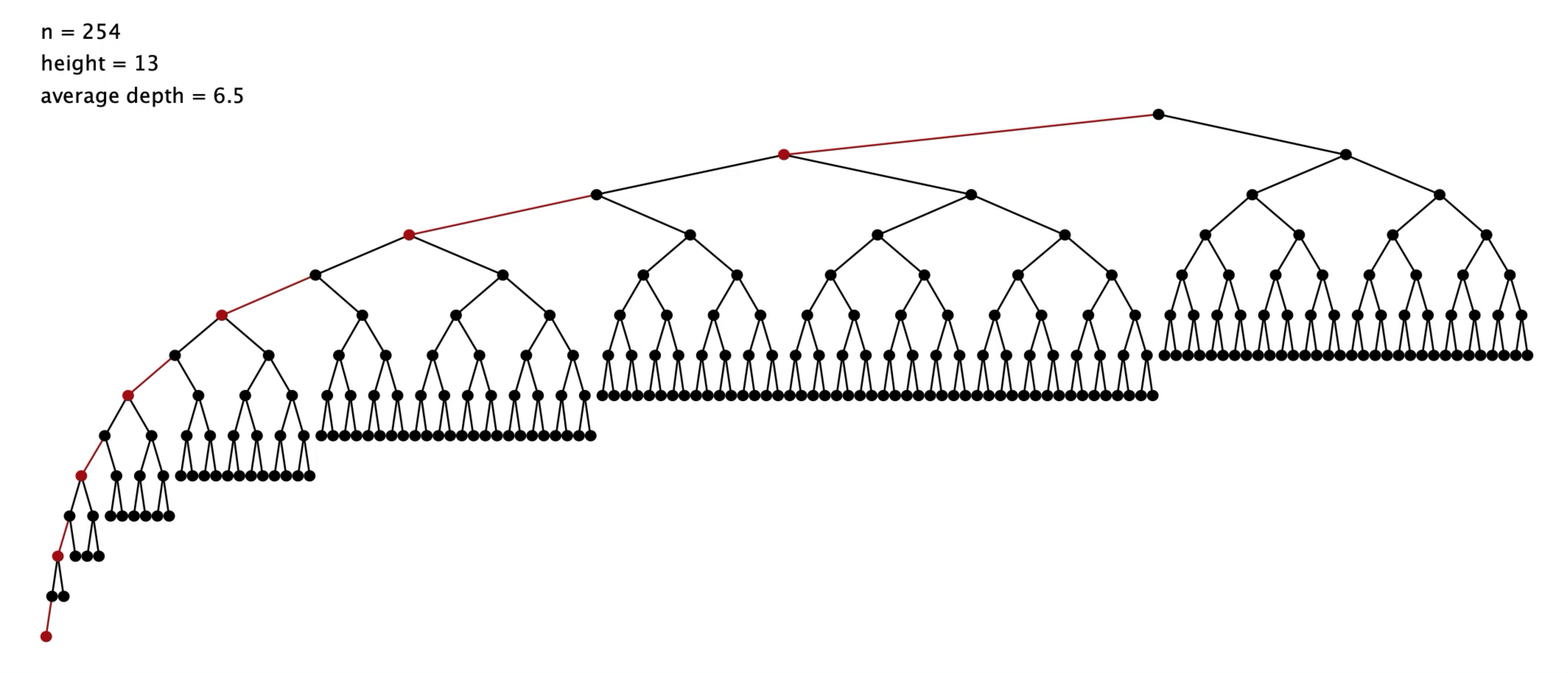
255 insertions in random order

Insertion into a LLRB tree: visualization



255 insertions in ascending order

Insertion into a LLRB tree: visualization



254 insertions in descending order

Balance in LLRB trees

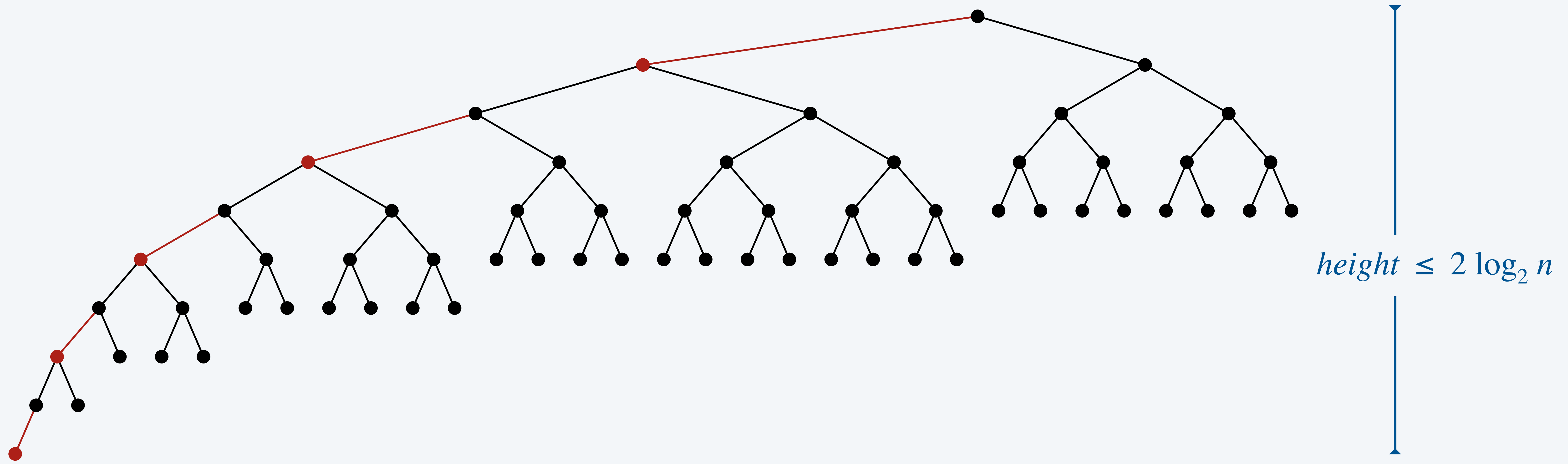
Proposition. Height of LLRB tree is $\leq 2 \log_2 n$.

Pf.

- Black height = height of corresponding 2–3 tree $\leq \log_2 n$.
- Never two red links in a row.

  $\Rightarrow$ height of LLRB tree $\leq (2 \times \textit{black height}) + 1$

  $\leq 2 \log_2 n + 1$.
- [A slightly more careful argument shows height $\leq 2 \log_2 n$.]



ST implementations: summary

implementation	guarantee			ordered ops?	key interface	emoji
	search	insert	delete			
sequential search (unordered list)	n	n	n		`equals()`	
binary search (sorted array)	$\log n$	n	n	✔	`compareTo()`	
BST	n	n	n	✔	`compareTo()`	
2-3 trees	$\log n$	$\log n$	$\log n$	✔	`compareTo()`	
red-black BSTs	$\log n$	$\log n$	$\log n$	✔	`compareTo()`	

hidden constant c is small
($\leq 2 \log_2 n$ *compares*)

https://algs4.cs.princeton.edu

3.3 Balanced Search Trees

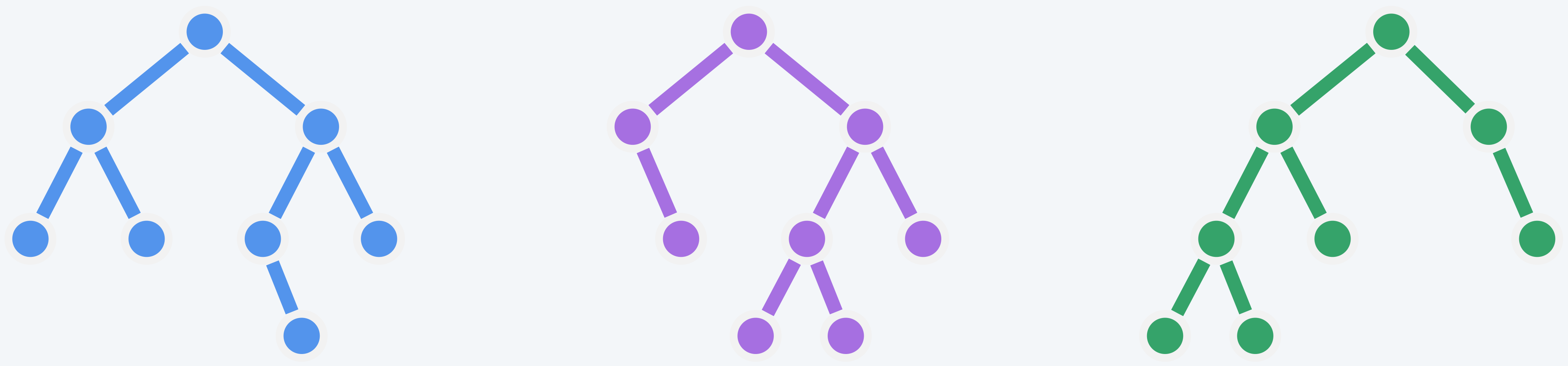
- *2–3 search trees*
- *red–black BSTs (representation)*
- *red–black BSTs (operations)*
- *context*

Balanced search trees in the wild

Red-black BSTs are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: CFQ I/O scheduler, VMAs, `linux/rbtree.h`.

Other balanced BSTs. AVL trees, splay trees, randomized BSTs, rank-balanced BSTs, ….



B-trees (and cousins) are widely used for file systems and databases.

Industry story 1: red-black BSTs

Telephone company contracted with database provider to build a real-time database to store customer information.

Database implementation.

- Red-black BST.
- Exceeding height limit of 80 triggered error-recovery process.

should support up to 2^{40} keys

Database crashed.

- Main cause = height bound exceeded!
- Telephone company sues database provider.
- Legal testimony:

> " *If implemented properly, the height of a red–black BST with n keys is at most* $2 \log_2 n$*.* " — expert witness


[redacted]

Industry story 2: red–black BSTs


Celestine Omin @cyberomin
Follow

I was just asked to balance a Binary Search Tree by JFK's airport immigration. Welcome to America.

8:26 AM - 26 Feb 2017 from Manhattan, NY

8,025 Retweets 7,087 Likes

Celestine Omin @cyberomin · 26 Feb 2017
I was too tired to even think of a BST solution. I have e been travelling for 23hrs. But I was also asked about 10 CS questions.
8 164 244

Celestine Omin @cyberomin · 26 Feb 2017
sad thing is, if I didn't give the Wikipedia definition for these questions, it was considered a wrong answer.
19 324 703

Simon Sharwood @ssharwood · 26 Feb 2017
Replying to @cyberomin
seriously? am reporter for @theregister and would love to know more about your experience
2 22 171

https://twitter.com/cyberomin/status/835888786462625792

The red–black tree song (by Sean Sandys)

Credits

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Red–Black Tree Song	Sean Sandys	by author
Red–Black Tree Song Video	U. Washington CSE Band	