COS 226	Algorithms and Data Structures	Fall 2023

Final Solutions

1. **Initialization.**

   Don't forget to do this.

2. **Empirical running time.**

   16,000, $\Theta(V^2 E)$

   When V *doubles, the running time goes up by a factor of* 4*, so the exponent for* V *is* $\log_2 4 = 2$*. When* E *quadruples, the running time goes up by a factor of* 4*, so the exponent for* E *is* $\log_4 4 = 1$*. Thus, the order of growth of the running time is* $\Theta(V^2 E)$*.*

3. **Depth-first search.**

   (a) 0 2 5 8 1 3 7 9 6 4

   (b) 5 1 7 9 3 6 8 2 4 0

   (c) yes
   The digraph is a DAG. So, the reverse postorder provides a topological order.

4. **Minimum spanning trees.**

   (a) 0 10 20 30 50 60 110

   (b) 30 0 20 50 60 10 110

5. **Shortest paths.**

   (a) 0 7 58 13 3 1

   (b) 0 4 5

6. **Maxflows and mincuts.**

   (a) $31 = 8 + 5 + 18$

   (b) $34 = 13 + 21$

   (c) 31
   The net flow across any cut is equal to the value of the flow.

   (d) $A \to F \to B \to G \to H$

   (e) 3
   The edge $B \to G$ is the bottleneck.

7. **Data structures.**

   (a) F *C would not be inserted at index 3 with index 0 empty.*
   T *The second one would arise if the keys were inserted in the order A B C D E.*
   T *The third one would arise if the keys were inserted in the order B A E D C.*

   (b) (10, 10), (12, 9)
   The constraints of the 2d-tree imply that, for any point (x, y) in T, we must have both $9 \leq x < 13$ and $8 \leq y < 14$.

   (c) $\Theta(n^2)$, $\Theta(n)$
   In the worst case (repeatedly removing the first element), each call to `remove()` *takes time proportional to number of elements remaining. This leads to a running time of $n + (n - 1) + \ldots 1$, which is $\Theta(n^2)$.*

   In the best case (repeatedly removing the last element), each call to `remove()` *takes $\Theta(1)$ time. Also, each call to* `append()` *takes $\Theta(1)$ amortized time. So, the overall running time is $\Theta(n)$.*

8. **Dynamic programming.**

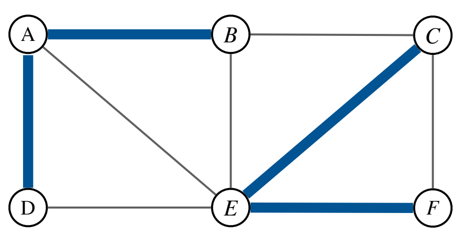
A C E H H L or C A E H H L

```
int[][] opt = new int[m+1][n+1];
for (int i = 1; i <= m; i++) {
    for (int j = 1; j <= n; j++) {
    if (times[j] > i) {
        opt[i][j] = opt[i][j-1];
    }
    else {
        opt[i][j] = Math.max(opt[i][j-1], points[j] + opt[i - times[j]][j-1]]);
    }
}
```

9. **Karger's algorithm.**

(a) A B D

It runs Kruskal's algorithm (using the random edge weights), adding the edges C—E, E—F, A—D, and A—B to T, until T contains exactly two connected components.



(b) 4

The edges that cross the cut are A—E, B—C, B—E, and D—E.

10. **Multiplicative weights.**

T F F F F F

11. **Intractability.**

N Y Y Y Y Y Y N

12. **Princeton path game.**

(a) To determine whether the orange player has already won:

- Build an edge-weighted graph G' containing only the orange edges.
- Run BFS (or DFS) to determine whether there is a directed path from s to t in G'.
- If such a path exists, declare orange the winner.

(b) To determine whether the black player has already won:

- Build an edge-weighted graph G'' containing the orange and uncolored edges (but not the black edges).
- Run BFS (or DFS) to determine whether there is a directed path from s to t in G''.
- If no such path exists, declare black the winner.

(c) The game cannot end in a tie.

Let's suppose the game continues until all edges are colored either orange or black. We'll see that exactly one player must win.

- *If there is a directed path P from s to t containing only orange edges, then orange wins (and black cannot simultaneously win because there are no black edges in P).*
- *Otherwise, consider the subset of vertices S reachable from s via orange edges, and let T be the remaining vertices. Note that $s \in S$ and $t \in T$. All edges that go from S to T are black and every directed path from s to t must use one (or more) of these edges. Thus, black wins.*

13. **Princeton minimum spanning trees.**

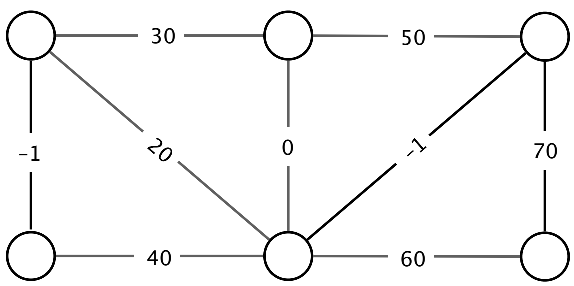
The main idea is to change the weight of all of the orange edges to a small value, smaller than the weight of any of the black edges. That way, the MST will prefer the orange edges to the black edges.

Step 1. Construct G':

- Create an edge-weighted graph G' that has the same vertices and edges as G.
- Let $w(e)$ and $w'(e)$ denote the weight of edge e in G and G', respectively.
- If edge e is black, set $w'(e) = w(e)$.
- If edge e is orange, set $w'(e) = \min_e w(e) - 1$.

Step 2. Compute the MST T' of G' via Prim or Kruskal.

- If T' contains all of the orange edges, then return T'.
- Otherwise, report *no Princeton-MST exists.*



Alternate solution (to determine whether Princeton MST exists).

- Create a graph G'' containing all of the orange edges in G.
- Determine whether G'' contains a cycle using DFS.
- If G'' contains a cycle, then report *no Princeton-MST exists.*

Alternate solution (to find MST). Create a graph G' formed by *contracting* all of the orange edges in G; compute the MST in G'; and return the corresponding edges in G. Some care is needed to contract the edges efficiently, which we won't describe here.